

Planar graphs with $\Delta \geq 8$ are $(\Delta + 1)$ -edge-choosable.*

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Abstract

We consider the problem of *list edge coloring* for planar graphs. Edge coloring is the problem of coloring the edges while ensuring that two edges that are incident receive different colors. A graph is k -edge-choosable if for any assignment of k colors to every edge, there is an edge coloring such that the color of every edge belongs to its color assignment. Vizing conjectured in 1965 that every graph is $(\Delta + 1)$ -edge-choosable. In 1990, Borodin solved the conjecture for planar graphs with maximum degree $\Delta \geq 9$, and asked whether the bound could be lowered to 8. We prove here that planar graphs with $\Delta \geq 8$ are $(\Delta + 1)$ -edge-choosable.

1 Introduction

We consider simple graphs. A k -edge-coloring of a graph G is a coloring of the edges of G with k colors such that two edges that are incident receive distinct colors. We denote by $\chi'(G)$ the smallest k such that G admits a k -edge-coloring. Let $\Delta(G)$ be the maximum degree of G . Since incident edges have to receive distinct colors in an edge coloring, every graph G verifies $\chi'(G) \geq \Delta(G)$. A trivial upper-bound on $\chi'(G)$ is $2\Delta(G) - 1$, which can be greatly improved, as follows.

Theorem 1 (Vizing [11]). *Every graph G verifies $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

Vizing [12] proved that $\chi'(G) = \Delta(G)$ for every planar graph G with $\Delta(G) \geq 8$. He gave examples of planar graphs with $\Delta(G) = 4, 5$ that are not $\Delta(G)$ -edge-colorable, and conjectured that no such graph exists for $\Delta(G) = 6, 7$. This remains open for $\Delta(G) = 6$, but the case $\Delta(G) = 7$ was solved by Sanders and Zhao [9], as follows.

Theorem 2 (Sanders and Zhao [9]). *Every planar graph G with $\Delta(G) \geq 7$ verifies $\chi'(G) = \Delta(G)$.*

An extension of the problem of edge coloring is the *list edge coloring* problem, defined as follows. For any $L : E \rightarrow \mathcal{P}(\mathbb{N})$ list assignment of colors to the edges of a graph $G = (V, E)$, the graph G is L -edge-colorable if there exists an edge coloring of G such that the color of every edge $e \in E$ belongs to $L(e)$. A graph $G = (V, E)$ is said to be *list k -edge-colorable* (or *k -edge-choosable*) if G is L -edge-colorable for any list assignment L such that $|L(e)| \geq k$ for any edge $e \in E$. We denote

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by $\chi'_\ell(G)$ the smallest k such that G is k -edge-choosable.

One can note that edge coloring is a special case of list edge coloring, where all the lists are equal. Thus $\chi'(G) \leq \chi'_\ell(G)$. This inequality is in fact conjectured to be an equality (see [7] for more information).

Conjecture 1 (List Coloring Conjecture). *Every graph G verifies $\chi'(G) = \chi'_\ell(G)$.*

The conjecture is still widely open. Some partial results were however obtained in the special case of planar graphs: for example, the conjecture is true for planar graphs of maximum degree at least 12, as follows.

Theorem 3 (Borodin et al. [5]). *Every planar graph G with $\Delta(G) \geq 12$ verifies $\chi'_\ell(G) = \Delta(G)$.*

There is still a large gap with the lower bound of 7 that should hold by Theorem 2 if Conjecture 1 were true.

Using Vizing's theorem, the List Coloring Conjecture can be weakened into Conjecture 2.

Conjecture 2 (Vizing [13]). *Every graph G verifies $\chi'_\ell(G) \leq \Delta(G) + 1$.*

Conjecture 2 has been actively studied in the case of planar graphs with some restrictions on cycles (see for example [10, 14, 15]), and was settled by Borodin [3] for planar graphs of maximum degree at least 9 (a simpler proof was later found by Cohen and Havet [6]).

Theorem 4 (Borodin [3]). *Every planar graph G with $\Delta(G) \geq 9$ verifies $\chi'_\ell(G) \leq \Delta(G) + 1$.*

Here we prove the following theorem.

Theorem 5. *Every planar graph G with $\Delta(G) \leq 8$ verifies $\chi'_\ell(G) \leq 9$.*

This improves Theorem 4 and settles Conjecture 2 for planar graphs of maximum degree 8.

Corollary 1. *Every planar graph G with $\Delta(G) \geq 8$ verifies $\chi'_\ell(G) \leq \Delta(G) + 1$.*

This answers Problem 5.9 in a survey by Borodin [4]. For small values of Δ , Theorem 5 implies that every planar graph G with $5 \leq \Delta(G) \leq 7$ is also 9-edge-choosable. To our knowledge, this was not known. It is however known that planar graphs with $\Delta(G) \leq 4$ are $(\Delta(G) + 1)$ -edge-choosable [8, 13].

In Sections 2 and 3, we introduce the method and some terminology. In Sections 4 to 6, we prove Theorem 5, with a discharging method.

2 Method

The discharging method was introduced in the beginning of the 20th century. It has been used to prove the celebrated Four Color Theorem ([1] and [2]).

We prove Theorem 5 using a discharging method, as follows. A graph is *minimal* for a property if it satisfies this property but none of its proper subgraphs does. The first step is to consider a minimal counter-example G (i.e. a graph G such that $\Delta(G) \leq 8$ and $\chi'_\ell(G) > 9$, whose every proper subgraph is 9-edge-choosable), and prove it cannot contain some configurations. We assume by contradiction that G contains one of the configurations. We consider a particular subgraph H of

G . For any list assignment L on the edges of G , with $|L(e)| \geq 9$ for every edge e , we L -edge-color H by minimality. We show how to extend the L -edge-coloring of H to G , a contradiction.

The second step is to prove that a connected planar graph on at least two vertices with $\Delta \leq 8$ that does not contain any of these configurations does not verify Euler's Formula. To that purpose, we consider a planar embedding of the graph. We assign to each vertex its degree minus six as a weight, and to each face two times its degree minus six. We apply discharging rules to redistribute weights along the graph with conservation of the total weight. As some configurations are forbidden, we can prove that after application of the discharging rules, every vertex and every face has a non-negative final weight. This implies that $\sum_v (d(v) - 6) + \sum_f (2d(f) - 6) = 2 \times |E(G)| - 6 \times |V(G)| + 4 \times |E(G)| - 6 \times |F(G)| \geq 0$, a contradiction with Euler's Formula that $|E| - |V| - |F| = -2$. Hence a minimal counter-example cannot exist.

3 Terminology and notation

In the figures, we draw in black a vertex that has no other neighbor than the ones already represented, in white a vertex that might have other neighbors than the ones represented. When there is a label inside a white vertex, it is an indication on the number of neighbors it has. The label ' i ' means "exactly i neighbors", the label ' i^+ ' (resp. ' i^- ') means that it has at least (resp. at most) i neighbors. Note that the white vertices may coincide with other vertices of the figure. The figures are only here to support the text and are not self-sufficient: the embedding can be different from the one represented.

For a given planar embedding, a vertex v is a *weak* neighbor of a vertex u when the two faces adjacent to the edge (u, v) are triangles (see Figure 1). A vertex v is a *semi-weak* neighbor of a vertex u when one of the two faces adjacent to the edge (u, v) is a triangle and the other is a cycle of length four (see Figure 2).

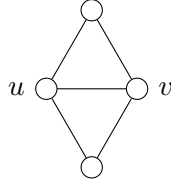


Figure 1: Vertex v is a weak neighbor of u .

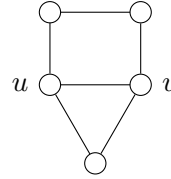


Figure 2: Vertex v is a semi-weak neighbor of u .

For any vertex u , we define special types of weak neighbors of degree 5 of u , as follows. The notation comes from E for "Eight" (when $d(u) = 8$) and S for "Seven" (when $d(u) = 7$). The index corresponds to the discharging rules (introduced in Section 5). Consider a weak neighbor v of degree 5 of u .

- Vertex v is an E_2 -neighbor of u with $d(u) = 8$ when one of the two following conditions is verified (see Figure 3):
 - There are two vertices w_1 and w_2 with $d(w_1) = d(w_2) = 6$ such that (u, v, w_1) and (v, w_1, w_2) are faces (see Figure 3a).
 - There are three vertices w_1, w_2 and w_3 with $d(w_1) = d(w_3) = 6$ and $d(w_2) = 7$ such that (u, v, w_1) , (v, w_1, w_2) and (u, v, w_3) are faces (see Figure 3b).

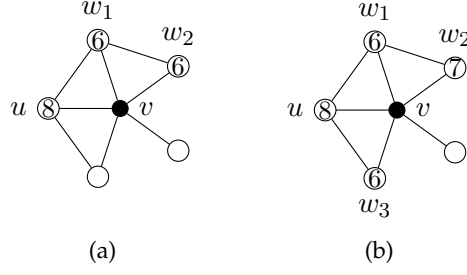


Figure 3: Vertex v is an E_2 -neighbor of u .

- Vertex v is an E_3 -neighbor of u with $d(u) = 8$ when v is not an E_2 -neighbor of u , and there is a vertex w with $d(w) \leq 7$ such that (u, v, w) is a face (see Figure 4).

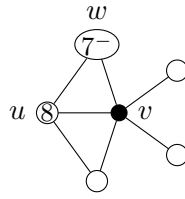


Figure 4: Vertex v is an E_3 -neighbor of u .

- Vertex v is an E_4 -neighbor of u with $d(u) = 8$ when v is not an E_2 nor an E_3 -neighbor of u (see Figure 5). That is, when the third vertices of the two faces containing the edge (u, v) are both of degree 8.

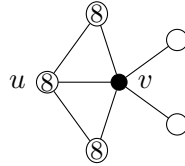


Figure 5: Vertex v is an E_4 -neighbor of u .

- Vertex v is an S_2 -neighbor of u with $d(u) = 7$ when there are two vertices w_1 and w_2 with $d(w_1) = d(w_2) = 6$ such that (u, v, w_1) and (u, v, w_2) are faces (see Figure 6).

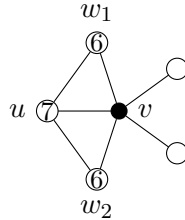


Figure 6: Vertex v is an S_2 -neighbor of u .

- Vertex v is an S_3 -neighbor of u with $d(u) = 7$ when v is not an S_2 -neighbor of u , and v has four neighbors w_1, w_2, w_3 and w_4 such that (u, v, w_1) and (u, v, w_4) are faces, and one of the following two conditions is verified:
 - $(v, w_1, w_2), (v, w_2, w_3)$ and (v, w_3, w_4) are faces, and $d(w_1) = d(w_4) = 7$ and $d(w_2) = d(w_3) = 6$ (see Figure 7a).
 - $d(w_4) = d(w_2) = 6$ and either $d(w_1) = 7$ (see Figure 7b) or $d(w_3) = 7$ (see Figure 7c).
Note that there is no constraint on the order of w_2 and w_3 in the embedding.

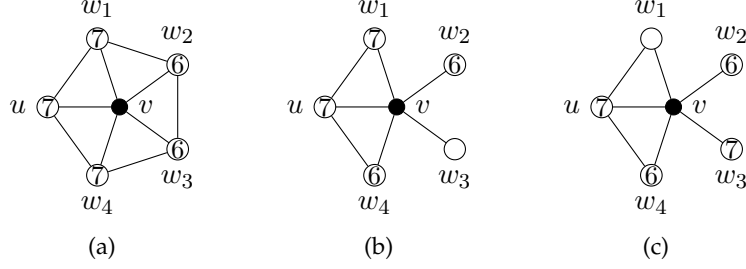


Figure 7: Vertex v is an S_3 -neighbor of u .

- Vertex v is an S_4 -neighbor of u with $d(u) = 7$ when v is not an S_2 - nor S_3 - neighbor of u , and either there is a vertex w with $d(w) \leq 7$ such that (u, v, w) is a face (see Figure 8a), or v is adjacent to two vertices w_1 and w_2 (both distinct from u) such that $d(w_1) = 6$ and $d(w_2) = 7$ (see Figure 8b).

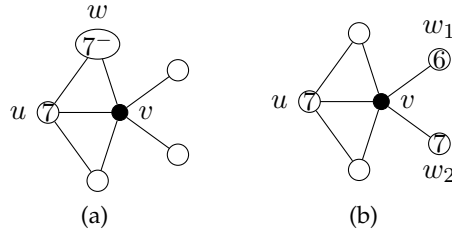


Figure 8: Vertex v is an S_4 -neighbor of u .

Note that if a weak neighbor v of u has none of the previous types, then u and v must verify one of the following four hypotheses:

- $d(v) \neq 5$
- $d(u) \notin \{7; 8\}$
- $d(u) = 7$, the third vertex of each of the two triangles adjacent to (u, v) is of degree 8, and the two other neighbors of v are either both of degree 6 or both of degree at least 7.

4 Forbidden Configurations

We define configurations (C_1) to (C_{11}) (see Figure 9).

- (C_1) is an edge (u, v) with $d(u) + d(v) \leq 10$.
- (C_2) is a cycle (u, v, w, x) such that $d(u) = d(w) = 3$.
- (C_3) is a vertex u with $d(u) = 8$ that has three neighbors v_1, v_2 and v_3 such that v_1 and v_2 are weak neighbors of u , with $d(v_1) = d(v_2) = 3$ and $d(v_3) \leq 5$.
- (C_4) is a vertex u with $d(u) = 8$ that has four neighbors v_1, v_2, v_3 and v_4 such that v_1 is a weak neighbor of u and v_2 is a semi-weak neighbor of u , with $d(v_1) = d(v_2) = 3$, $d(v_3) \leq 5$ and $d(v_4) \leq 5$.
- (C_5) is a vertex u with $d(u) = 8$ that has four weak neighbors v_1, v_2, v_3 and v_4 with $d(v_1) = 3$, $d(v_2) = d(v_3) = 4$ and $d(v_4) \leq 5$.
- (C_6) is a vertex u with $d(u) = 8$ that has five neighbors v_1, v_2, v_3, v_4 and v_5 such that v_1 is a weak neighbor of u with $d(v_1) = 3$, $d(v_2) = 4$, $d(v_3) \leq 5$, $d(v_4) \leq 5$ and $d(v_5) \leq 7$.
- (C_7) is a vertex u with $d(u) = 8$ that has four weak neighbors v_1, v_2, v_3 and v_4 , such that $d(v_1) = 3$, vertex v_2 is an E_2 -neighbor of u , $d(v_3) \leq 5$ and $d(v_4) \leq 5$.
- (C_8) is a vertex u with $d(u) = 7$ that has three neighbors v, w and x such that w is adjacent to v and x , $d(w) = 6$, $d(v) = d(x) = 5$, and there is a vertex y of degree 6, distinct from w , that is adjacent to x .
- (C_9) is a vertex u with $d(u) = 7$ that has three weak neighbors v_1, v_2 and v_3 such that $d(v_1) = d(v_2) = 4$ and either v_3 is an S_2, S_3 or S_4 -neighbor, or $d(v_3) = 4$.
- (C_{10}) is a vertex u with $d(u) = 7$ that has three neighbors v_1, v_2 and v_3 such that $d(v_1) = 4$, vertex v_2 is an S_3 -neighbor of u and $d(v_3) \leq 5$.
- (C_{11}) is a vertex u with $d(u) = 5$ that has three neighbors v, w and x such that w is adjacent to v and x , and $d(v) = d(w) = d(x) = 6$.

We first introduce the three following useful lemmas.

Lemma 1. [13] Every even cycle C verifies $\chi'_\ell(C) = 2$.

Lemma 2. Let G be the graph with five edges (a, b, c, d, e) such that (b, c, d, e) forms a cycle and a is incident only to b and e (see Figure 10). Let $L : \{a, b, c, d, e\} \rightarrow \mathcal{P}(\mathbb{N})$ a list assignment of at least two colors on every edge, where either $|L(b)| \geq 3$ or $L(b) \neq L(a)$. The graph G is L -edge-colorable.

Proof. We consider w.l.o.g. the worst case, i.e. $|L(a)| = |L(c)| = |L(d)| = |L(e)| = 2$. We consider two cases depending on whether $L(c) \cap L(e) = \emptyset$.

- $L(c) \cap L(e) \neq \emptyset$.
Then let $\alpha \in L(c) \cap L(e)$. We color c and e in α . Since $|L(b)| \geq 3$ or $L(a) \neq L(b)$, we can color a and b . We color d .

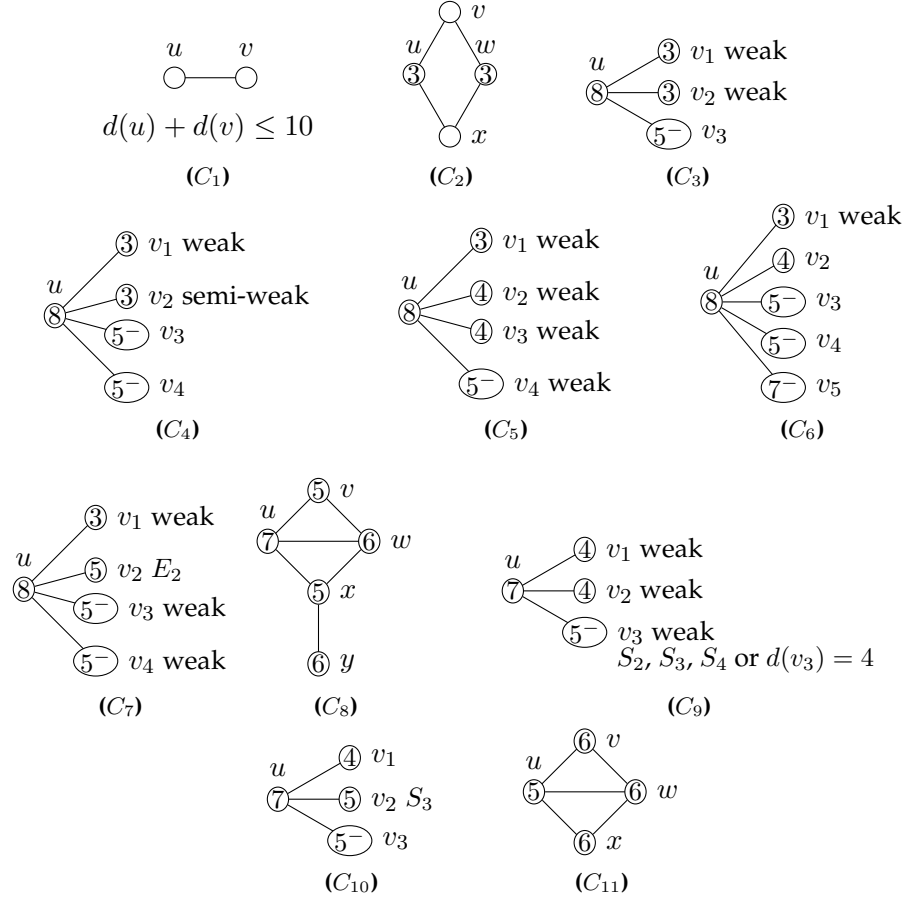


Figure 9: Forbidden configurations.

- $L(c) \cap L(e) = \emptyset$.
Then let $\alpha \in L(b) \setminus L(a)$. We color b in α , and consider two cases depending on whether $\alpha \notin L(c)$ or $\alpha \notin L(e)$.
 - $\alpha \notin L(c)$.
Then we color successively e, a, d and c .
 - $\alpha \notin L(e)$.
Then we color successively c, d, e and a .

□

Lemma 3. Let G be the star on three edges (a, b, c) . Let $L : \{a, b, c\} \rightarrow \mathcal{P}(\mathbb{N})$ a list assignment such that $|L(a)| \geq 2$, $|L(b)| \geq 2$, $|L(c)| \geq 2$. The graph G is L -edge-colorable unless $L(a)$, $L(b)$ and $L(c)$ are all equal and of cardinality 2.

Proof. Assume we do not have $L(a) = L(b) = L(c)$ with $|L(a)| = 2$. We assume without loss of generality that $|L(a)| \geq 3$ or that $|L(a)| = |L(b)| = |L(c)| = 2$ with $L(a) \neq L(b)$ and $L(a) \neq L(c)$. We color c in a color that is not available for a if possible, in an arbitrary color otherwise. We color successively b and a . □

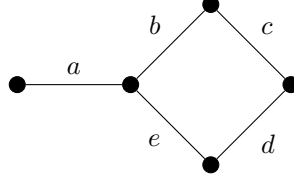


Figure 10: The graph of Lemma 2.

Lemma 4. *If G is a minimal planar graph with $\Delta(G) \leq 8$ such that $\chi'_\ell(G) > 9$, then G does not contain any of Configurations (C_1) to (C_{11}) .*

Proof. Let L be a list assignment on the edges of G with $|L(e)| \geq 9$ for every edge e of G . We prove that if G contains any of Configurations (C_1) to (C_{11}) , then there is a subgraph H of G , that can be L -edge-colored by minimality, and whose L -edge-coloring is extendable to G , a contradiction.

A *constraint* of an edge $e \in E$ is an already colored edge that is incident to e . In the following, we denote generically \hat{e} the list of available colors for an edge e at the moment it is used: the list is implicitly modified as incident edges are colored. Proving that the L -edge-coloring of H can be extended to G is equivalent to proving that the graph induced by the edges that are not colored yet is L' -colorable, where $L'(e) = \hat{e}$ for every edge e . We sometimes *delete* edges. Deleting an edge means that no matter the coloring of the other uncolored edges, there will still be a free color for it (for example, when the edge has more colors available than uncolored incident edges). Thus the deleted edge is implicitly colored after the remaining uncolored edges.

We use the same notations as in the definition of Configurations (C_1) to (C_{11}) (see Figure 9).

Claim 1. *G cannot contain (C_1) .*

Proof. Using the minimality of G , we color $G \setminus \{(u, v)\}$. Since $d(u) + d(v) \leq 10$, the edge (u, v) has at most $10 - 2$ constraints. There are 9 colors, so we can color (u, v) , thus extending the coloring to G . \square

Claim 2. *G cannot contain (C_2) .*

Proof. Using the minimality of G , we color $G \setminus \{(u, v), (v, w), (w, x), (x, u)\}$. Since $\Delta(G) \leq 8$ and $d(u) = d(w) = 3$, every uncolored edge has at most $8 - 2 + 1$ constraints. There are 9 colors, so every uncolored edge has at least two available colors, and they form a cycle of length four. We can thus apply Lemma 1 to extend the coloring to G . \square

Claim 3. *G cannot contain (C_3) .*

Proof. By Claim 2, vertices v_1 and v_2 have no common neighbor other than u . By Claim 1, for $i \in \{1, 2, 3\}$, vertex v_i is adjacent only to vertices of degree at least 6. So the v_i 's are pairwise non-adjacent. We name the edges according to Figure 11.

By minimality of G , we color $G \setminus \{v_1, v_2\}$. Since there are 9 colors and every vertex is of degree at most 8, we have $|\hat{a}_1|, |\hat{a}_2|, |\hat{b}_1|, |\hat{b}_2| \geq 2$ and $|\hat{c}_1|, |\hat{c}_2| \geq 3$. We first prove the following.

- (1) *If $\hat{a}_1 = \hat{b}_1$, $\hat{a}_2 = \hat{b}_2$, and $|\hat{a}_1| = |\hat{b}_1| = |\hat{a}_2| = |\hat{b}_2| = 2$. Then we can recolor $G \setminus \{v_1, v_2\}$ so that the hypothesis is not verified anymore.*

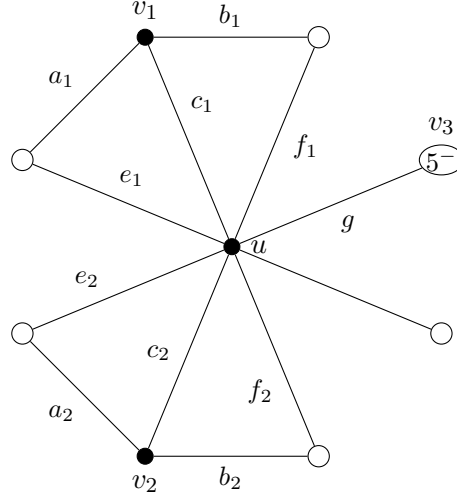


Figure 11: Notations of Claim 3

Proof. For $i \in \{1, 2\}$, let $\hat{a}_i = \hat{b}_i = \{\alpha_i, \beta_i\}$. For $i \in \{1, 2\}$, let γ_i and δ_i be the color of e_i and f_i , respectively. Note that $\gamma_i \in L(a_i)$ and $\delta_i \in L(b_i)$ since $|\hat{a}_i| = |\hat{b}_i| = 2$. Note that for a given $i \in \{1, 2\}$, the colors $\alpha_i, \beta_i, \gamma_i$ and δ_i are all different.

We claim that any recoloring of $\{e_1, f_1, e_2, f_2, g\}$ such that the color of at least one of $\{e_1, f_1, e_2, f_2\}$ has been changed breaks the hypothesis of (1). Indeed, assume w.l.o.g. that the color of e_1 can be changed while recoloring only edges of $\{e_1, f_1, e_2, f_2, g\}$, and consider such a coloring. We have $\gamma_1 \in \hat{a}_1$ since $\gamma_1 \in L(a_1)$ and the only edge of $\{e_1, f_1, e_2, f_2, g\}$ that is incident to a_1 is e_1 , which is not colored in γ_1 anymore. We have $\gamma_1 \notin \hat{b}_1$ since $\gamma_1 \notin \{\alpha_1, \beta_1, \delta_1\}$ and the only edge of $\{e_1, f_1, e_2, f_2, g\}$ that is incident to b_1 is f_1 , which was colored in δ_1 . Thus $\hat{a}_1 \neq \hat{b}_1$, and the hypothesis of (1) is broken.

We prove now that there exists such a recoloring. Aside from the constraints derived from $\{e_1, e_2, f_1, f_2, g\}$, each edge e_i or f_i has at most $(8 - 2) + (8 - 7) = 7$ constraints, and g has at most $(5 - 1) + (8 - 7) = 5$ constraints. Let L' be the list assignment of the colors available for those edges, when ignoring the constraints derived from $\{e_1, e_2, f_1, f_2, g\}$. Note that $|L'(e_i)|, |L'(f_i)| \geq 2$ and $|L'(g)| \geq 4$. Let us build the directed graph D whose vertex set is $V(D) = \{e_1, e_2, f_1, f_2, g\}$ and where for any two distinct $u, v \in V(D)$, there is an edge from u to v if the color of u belongs to $L'(v)$. We consider two cases depending on whether there is a cycle in D .

- *There is a cycle in D .*

Then we recolor accordingly the edges in G (for any edge from u to v in the cycle, v takes the initial color of u , which belongs by definition to $L'(v)$). Since a cycle contains at least two vertices, at least one of $\{e_1, e_2, f_1, f_2\}$ has been recolored.

- *There is no cycle in D .*

Then some vertex has in-degree 0. We consider two cases depending on whether some e_i or f_i has in-degree 0.

- *Some e_i or f_i has in-degree 0.*

Then it can be recolored without conflict (i.e. without recoloring the other vertices of D).

- Every e_i and f_i has in-degree at least 1.

Then g has in-degree 0. So g can be recolored without conflict. Since there is no cycle in D and every e_i and f_i is of in-degree at least 1, there is necessarily an edge from g to some e_i or f_i , which can now be recolored without conflict.

◇

By (1), we can assume that we have a coloring of $G \setminus \{v_1, v_2\}$ that does not verify the hypothesis of (1). W.l.o.g., we consider the case where $\hat{a}_1 \neq \hat{b}_1$ or $|\hat{a}_1| \geq 3$. We color a_2, b_2 and c_2 . Then $|\hat{c}_1| \geq 2$ and \hat{a}_1 and \hat{b}_1 have not been modified. So we apply Lemma 3 to the edges incident to v_1 . □

Claim 4. G cannot contain (C_4) .

Proof. We prove Claim 4 similarly as Claim 3. By Claim 2, vertices v_1 and v_2 have no common neighbor other than u . By Claim 1, for $i \in \{1, 2, 3, 4\}$, vertex v_i is adjacent only to vertices of degree at least 6. So the v_i 's are pairwise non-adjacent. We name the edges according to Figure 12. Note that among the edges named here, the edge b_2 is incident only to a_2 and c_2 .

By minimality of G , we color $G \setminus \{v_1, v_2\}$. Since there are 9 colors and every vertex is of degree at most 8, we have $|\hat{a}_1|, |\hat{a}_2|, |\hat{b}_1|, |\hat{b}_2| \geq 2$ and $|\hat{c}_1|, |\hat{c}_2| \geq 3$. We proceed as for Claim 3 and prove the following.

- (2) *If $\hat{a}_1 = \hat{b}_1$, $\hat{a}_2 = \hat{b}_2$, and $|\hat{a}_1| = |\hat{b}_1| = |\hat{a}_2| = |\hat{b}_2| = 2$. Then we can recolor $G \setminus \{v_1, v_2\}$ so that the hypothesis is not verified anymore.*

Proof. For $i \in \{1, 2\}$, let $\hat{a}_i = \hat{b}_i = \{\alpha_i, \beta_i\}$. For $i \in \{1, 2\}$, let γ_i be the color of e_i . Let δ_1 be the color of f_1 . Note that $\gamma_i \in L(a_i)$ since $|\hat{a}_i| = 2$. Similarly, $\delta_1 \in L(b_1)$. Note that for a given $i \in \{1, 2\}$, the colors $\alpha_i, \beta_i, \gamma_i$ (and δ_1 if $i = 1$) are all different.

We claim that any recoloring of $\{e_1, f_1, e_2, g_1, g_2\}$ such that the color of at least one of $\{e_1, f_1, e_2\}$ has been changed breaks the hypothesis of (2). Indeed, assume that the color of e_1 can be changed while recoloring only edges of $\{e_1, f_1, e_2, g_1, g_2\}$, and consider such a coloring. (The cases where the color of f_1 or e_2 can be changed are similar). We have $\gamma_1 \in \hat{a}_1$ since $\gamma_1 \in L(a_1)$ and the only edge of $\{e_1, f_1, e_2, g_1, g_2\}$ that is incident to a_1 is e_1 , which is not colored in γ_1 anymore. We have $\gamma_1 \notin \hat{b}_1$ since $\gamma_1 \notin \{\alpha_1, \beta_1, \delta_1\}$ and the only edge of $\{e_1, f_1, e_2, f_2, g\}$ that is incident to b_1 is f_1 , which was colored in δ_1 . Thus $\hat{a}_1 \neq \hat{b}_1$.

We prove now that there exists such a recoloring. Aside from the constraints derived from $\{e_1, e_2, f_1, g_1, g_2\}$, each edge e_1, e_2 and f_1 has at most $(8-2)+(8-7) = 7$ constraints, and each g_i has at most $4 + 1 = 5$ constraints. Let L' be the list assignment of the colors available for those edges, when ignoring the constraints derived from $\{e_1, e_2, f_1, g_1, g_2\}$. Note that $|L'(e_i)|, |L'(f_1)| \geq 2$, and $|L'(g_i)| \geq 4$. We consider w.l.o.g. the worst case, i.e. $|L'(e_1)| = |L'(e_2)| = |L'(f_1)| = 2$. Let us build the directed graph D whose vertex set is $V(D) = \{e_1, e_2, f_1, g_1, g_2\}$ and where there is an edge from u to v if the color of u belongs to $L'(v)$.

First note that if there is an edge from some g_i to some $v \in \{e_1, e_2, f_1\}$, then there are all edges from $\{e_1, e_2, f_1\} \setminus \{v\}$ to g_i . Indeed, if vertex g_i has in-degree at most 2, we recolor g_i and recolor v into the former color of g_i . So we assume vertex g_i has in-degree at least 3. If there is an edge from v to g_i , we exchange the colors of v and g_i . Thus there are all possible edges from $\{e_1, e_2, f_1\} \setminus \{v\}$ to g_1 .

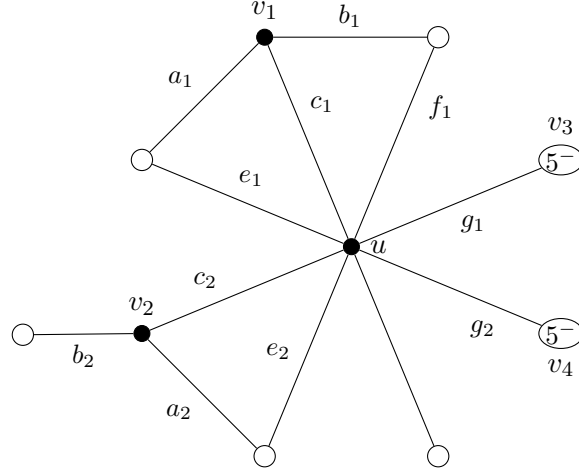


Figure 12: Notations of Claim 4

If some e_i or f_i has in-degree 0, we can recolor it without conflict. So we can assume that all of e_1, e_2 and f_1 have in-degree at least 1. If there is no edge from $\{g_1, g_2\}$ to $\{e_1, e_2, f_1\}$, then there is a directed cycle in $\{e_1, e_2, f_1\}$, and we recolor accordingly the edges in G . So there is at least an edge from $\{g_1, g_2\}$ to $\{e_1, e_2, f_1\}$. We consider w.l.o.g. that there is an edge from g_1 to e_1 . By the previous remark, there is an edge from e_2 and f_1 to g_1 . Both e_2 and f_1 have in-degree at least 1. If there is an edge from e_2 to f_1 and an edge from f_1 to e_2 , we exchange their colors. So we assume w.l.o.g. that there is an edge from $\{e_1, g_1, g_2\}$ to e_2 . If there is an edge from e_1 to e_2 , there is a directed cycle on $\{e_1, e_2, g_1\}$, and we recolor accordingly the edges in G . If there is an edge from g_1 to e_2 , we exchange the colors of g_1 and e_2 . If there is an edge from g_2 to e_2 , then by the previous remark, there is an edge from e_1 to g_2 . Thus there is a directed cycle on $\{e_1, g_2, e_2, g_1\}$ and we recolor accordingly the edges in G . \diamond

By (2), we can assume that we have a coloring of $G \setminus \{v_1, v_2\}$ that does not verify the hypothesis of (2). W.l.o.g., we consider the case where $\hat{a}_1 \neq \hat{b}_1$ or $|\hat{a}_1| \geq 3$. We color a_2, b_2, c_2 and apply Lemma 3 to the edges incident to v_1 . \square

Claim 5. G cannot contain (C_5) .

Proof. By Claim 1, no two v_i are adjacent. Since every v_i is a weak neighbor of u , and $d(u) = 8$, the neighborhood of u forms a cycle (see Figure 13). We consider two cases depending on whether there is a vertex x such that v_2, x and v_3 appear consecutively around u .

- There is a vertex x such that v_2, x and v_3 appear consecutively around u .

We consider without loss of generality that the neighbors of u are, clockwise, $v_1, w_1, v_2, w_2, v_3, w_3, v_4$ and w_4 . We name the edges according to Figure 13a. Note that the edges l and o are distinct. By minimality of G , we color $G \setminus \{a, \dots, r\}$.

Without loss of generality, we consider the worst case, i.e. $|\hat{l}| = |\hat{o}| = |\hat{q}| = |\hat{r}| = 2$, $|\hat{b}| = |\hat{d}| = |\hat{f}| = |\hat{h}| = |\hat{i}| = |\hat{j}| = |\hat{k}| = |\hat{m}| = |\hat{n}| = |\hat{p}| = 4$, $|\hat{g}| = 7$, and $|\hat{a}| = |\hat{c}| = |\hat{e}| = 9$. We consider two cases depending on whether $\hat{i} = \hat{j}$ and $\hat{i} \cap \hat{h} \neq \emptyset$.

- $\hat{i} \neq \hat{j}$ or $\hat{i} \cap \hat{h} = \emptyset$.
 If $\hat{i} \neq \hat{j}$, we color i in a color that does not belong to \hat{j} . Otherwise $\hat{h} \cap \hat{i} = \emptyset$ and i can be deleted. In any case $|\hat{j}| = 4$ and j has exactly 3 uncolored incident edges, so we can delete it. Then $|\hat{a}| \geq 8$ and a has 7 uncolored incident edges, so we can delete it. Since $|\hat{b}| + |\hat{l}| > |\hat{k}|$, there exists a color $\alpha \in (\hat{b} \cap \hat{l}) \cup ((\hat{b} \cup \hat{l}) \setminus \hat{k})$. Note that b and l are not incident. We color b and l in α if possible, in an arbitrary color otherwise. If $\alpha \in \hat{b} \cap \hat{l}$, then b and l are colored in α and $|\hat{k}| \geq 3$. If $\alpha \in (\hat{b} \cup \hat{l}) \setminus \hat{k}$, then at least one of b and l is colored in α and $|\hat{k}| \geq 3$. So we can delete k . Then, successively, $c, m, e, n, o, p, g, d, f, h, q$ and r can be deleted.
- $\hat{i} = \hat{j}$ and $\hat{i} \cap \hat{h} \neq \emptyset$.
 Then let $\alpha \in \hat{j} \cap \hat{h}$. Note that j and h are not incident. We color j and h in α . Since i (resp. a) is incident to both j and h , we can successively delete i and a . We color successively r and q . Without loss of generality, we consider the worst case, i.e. $|\hat{f}| = |\hat{l}| = |\hat{o}| = 2$, $|\hat{b}| = |\hat{d}| = |\hat{k}| = |\hat{p}| = 3$, $|\hat{g}| = |\hat{m}| = |\hat{n}| = 4$, and $|\hat{c}| = |\hat{e}| = 8$. We consider three cases depending on whether $\hat{f} \cap \hat{n} = \emptyset$ and $\hat{p} \setminus \hat{o} \subset \hat{n}$.
 - * $\hat{f} \cap \hat{n} \neq \emptyset$.
 Then let $\beta \in \hat{f} \cap \hat{n}$. We color f and n in β . We delete successively e, p, o, c , and g . we color l . We apply Lemma 1 on (b, k, m, d) .
 - * $\hat{p} \setminus \hat{o} \not\subset \hat{n}$.
 Then let $\beta \in \hat{p} \setminus (\hat{o} \cup \hat{n})$. We color p in β . We color f . Since $|\hat{m}| + |\hat{o}| > |\hat{n}|$, there exists $\gamma \in (\hat{m} \cap \hat{o}) \cup ((\hat{m} \cup \hat{o}) \setminus \hat{n})$. If $\gamma \in \hat{d}$ or $\gamma \notin \hat{m}$, we color d and o in γ if possible, in an arbitrary color otherwise. We delete successively n, e, c, m, l, k, g and b . If $\gamma \notin \hat{d}$ and $\gamma \in \hat{m}$, we color m and o in γ if possible, in an arbitrary color otherwise. Note that $|\hat{d}| \geq 2$. We delete successively e, c, g, d, b, k and l .
 - * $\hat{f} \cap \hat{n} = \emptyset$ and $\hat{p} \setminus \hat{o} \subset \hat{n}$.
 Then let $\beta \in \hat{p} \setminus \hat{o}$. We color p in β . By assumption, $\beta \notin \hat{f} \cup \hat{o}$. We color n in a color that does not belong to o . We delete successively o, e, c and g . We color l , and apply Lemma 2 on (f, b, k, m, d) .
- There is no vertex x such that v_2, x and v_3 appear consecutively around u .
 We consider without loss of generality that the neighbors of u are, clockwise, $v_1, w_1, v_2, w_2, v_4, w_3, v_3$ and w_4 . We name the edges according to Figure 13b. By minimality of G , we color $G \setminus \{a, \dots, r\}$.
 Without loss of generality, we consider the worst case, i.e. $|\hat{l}| = |\hat{n}| = |\hat{o}| = |\hat{q}| = 2$, $|\hat{b}| = |\hat{d}| = |\hat{f}| = |\hat{h}| = |\hat{i}| = |\hat{j}| = |\hat{k}| = |\hat{m}| = |\hat{p}| = |\hat{r}| = 4$, $|\hat{e}| = 7$, and $|\hat{a}| = |\hat{c}| = |\hat{g}| = 9$. We consider two cases depending on whether $\hat{i} = \hat{j}$ and $\hat{i} \cap \hat{h} \neq \emptyset$.
 - $\hat{i} \neq \hat{j}$ or $\hat{i} \cap \hat{h} = \emptyset$.
 If $\hat{i} \neq \hat{j}$, we color i in a color that does not belong to \hat{j} . Otherwise $\hat{i} \cap \hat{h} = \emptyset$, we can delete i . In both cases, we can delete successively j and a .
 We consider three cases depending on whether $\hat{q} \cap \hat{h} = \emptyset$ and $\hat{q} \subset \hat{r}$.
 - * $\hat{q} \cap \hat{h} \neq \emptyset$.
 Then let $\alpha \in \hat{q} \cap \hat{h}$. We color q and h in α . We delete successively $g, c, e, r, p, k, m, l, b, d, n, p$ and f .

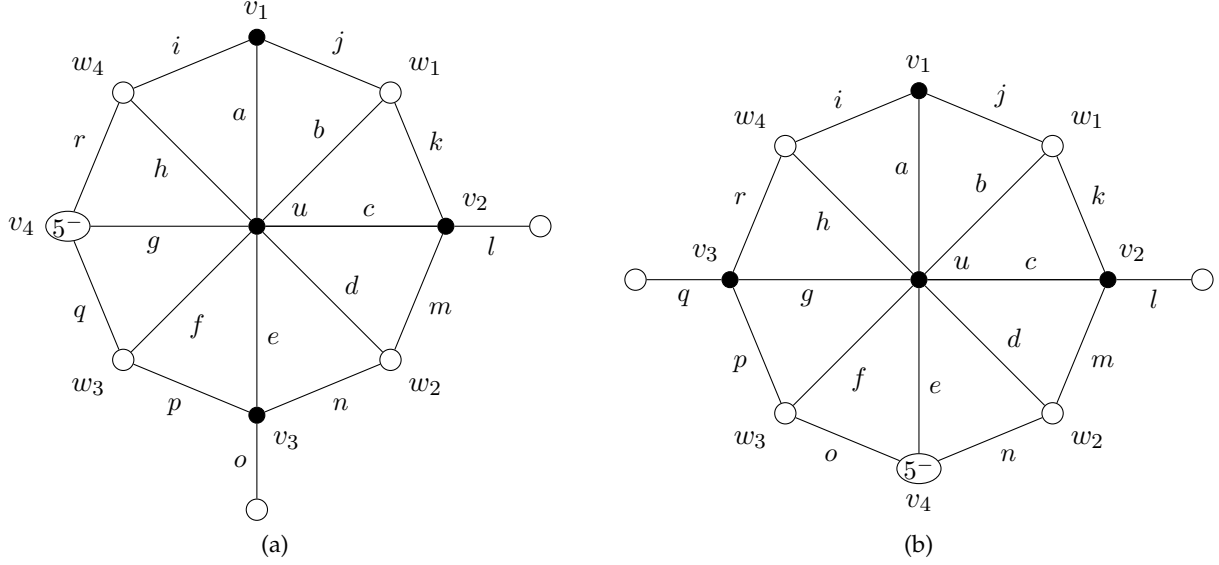


Figure 13: Notations of Claim 5

* $\hat{q} \not\subset \hat{r}$.

Then let $\alpha \in \hat{q} \setminus \hat{r}$. We color q in α . Since $|\hat{h}| + |\hat{p}| > |\hat{r}|$, there exists a color $\beta \in (\hat{h} \cap \hat{p}) \cup ((\hat{h} \cup \hat{p}) \setminus \hat{r})$. We color h and p in β if possible, in an arbitrary color otherwise. We delete successively $r, g, c, e, k, l, m, b, d, f, n$ and o .

* $\hat{q} \cap \hat{h} = \emptyset$ and $\hat{q} \subset \hat{r}$.

Then let $\alpha \in \hat{q}$. By assumption, $\alpha \in \hat{r} \setminus \hat{h}$. We color r in α , and color q . Since $|\hat{b}| + |\hat{l}| > |\hat{k}|$, there exists a color $\beta \in (\hat{b} \cap \hat{l}) \cup ((\hat{b} \cup \hat{l}) \setminus \hat{k})$. We color b and l in β if possible, in an arbitrary color otherwise. We delete successively k, c, g, e , and m . We color h in such a way that afterwards, $|\hat{f}| \geq 3$ or $\hat{f} \neq \hat{p}$. Then we apply Lemma 2 on (p, f, d, n, o) .

– $\hat{i} = \hat{j}$ and $\hat{i} \cap \hat{h} \neq \emptyset$.

Since $\hat{i} = \hat{j}$, there exists $\alpha \in \hat{j} \cap \hat{h}$, we color j and h in α , and delete i and a . When we say that we color $q|r$ in a color α , it means that we color q in α if possible, otherwise we color r in α .

Let $C = \hat{c}$ and $G = \hat{g}$. If $\hat{q} \cup \hat{r} \not\subset \hat{g}$, we consider $\alpha \in (\hat{q} \cup \hat{r}) \setminus \hat{g}$, and color $q|r$ in α . Assume that $\hat{q} \cup \hat{r} \subset \hat{g}$. Note that $|((\hat{q} \cup \hat{r}) \cap \hat{c}) \cup (\hat{c} \setminus \hat{g})| \geq |\hat{q} \cup \hat{r}| \geq 3$, and that $|\hat{l}| = 2$. We consider $\alpha \in (((\hat{q} \cup \hat{r}) \cap \hat{c}) \cup (\hat{c} \setminus \hat{g})) \setminus \hat{l}$. We color $q|r$ in α if possible, in an arbitrary color otherwise.

Note that since q and r have the same incidencies in the resulting graph, and since $|\hat{r}| \geq |\hat{q}| - 1$, the identity of the edge that is colored has no impact, and we can consider w.l.o.g. that q is colored and r remains uncolored. We remove color α from \hat{k} and \hat{m} . We consider w.l.o.g. the worst case, i.e. $|\hat{k}| = |\hat{l}| = |\hat{n}| = |\hat{o}| = |\hat{r}| = 2$, $|\hat{b}| = |\hat{d}| = |\hat{f}| = |\hat{m}| = |\hat{p}| = 3$, $|\hat{e}| = 6$, $|\hat{g}| = 7$ and $|\hat{c}| = 8$.

We consider two cases depending on whether $\hat{k} = \hat{l}$.

* $\hat{k} = \hat{l}$. Then we color m in a color that does not belong to \hat{l} . We color successively n, o, d, f, b, k and l .

- * $\hat{k} \neq \hat{l}$. Then we color l in a color that does not belong to \hat{k} . If $\hat{m} = \hat{n}$, then we color d in a color that does not belong to \hat{m} , and apply Lemma 1 on (b, k, m, n, o, f) . If $\hat{m} \neq \hat{n}$, then we color m in a color that does not belong to \hat{n} , we color k and we apply Lemma 2 on (b, f, o, n, d) .

We then color p, q and e . We claim that $\hat{c} \neq \hat{g}$ if $|\hat{c}| = |\hat{g}| = 1$. Indeed, assume $|\hat{c}| = |\hat{g}| = 1$. Then, all the edges incident to g are colored differently, and their colors belong to G . We consider two cases depending on whether q is colored in α .

- * *Edge q is colored in α .* Then $\alpha \in G$, which implies $\alpha \in C$ by choice of α . Since the edges incident to g are all colored differently and q is colored in α , none of $\{b, d, e, f, h\}$ is colored in α . By construction, none of $\{k, m\}$ is colored in α . By choice of α , l is not colored in α . Thus $\alpha \in \hat{c}$ and $\alpha \notin \hat{g}$, so $\hat{c} \neq \hat{g}$.
- * *Edge q is not colored in α .* Then, by choice of α , we have $\alpha \in C \setminus G$. Since the colors of the edges incident to g all belong to G , none of $\{b, d, e, f, h\}$ is colored in α . By construction, none of $\{k, m\}$ is colored in α . By choice of α , l is not colored in α . Thus $\alpha \in \hat{c}$ and $\alpha \notin \hat{g}$, so $\hat{c} \neq \hat{g}$.

Note that $|\hat{c}| \geq 1$ and $|\hat{g}| \geq 1$. If $|\hat{c}| = |\hat{g}| = 1$, then $\hat{c} \neq \hat{g}$, so we color c and g independently. If not, assume w.l.o.g. that $|\hat{c}| \geq 2$, and color successively g and c .

□

Claim 6. G cannot contain (C_6) .

Proof. We prove Claim 6 similarly as Claim 3. By Claim 1, for $i \in \{1, 2, 3, 4, 5\}$, vertex v_i is adjacent only to vertices of degree at least $11 - d(v_i)$. We name the edges according to Figure 14. By

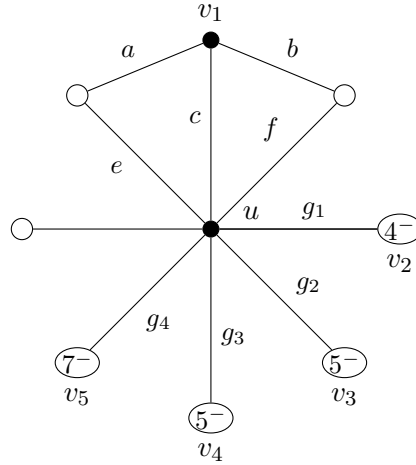


Figure 14: Notations of Claim 6

minimality of G , we color $G \setminus \{v_1\}$. Since there are 9 colors and every vertex is of degree at most 8, we have $|\hat{a}|, |\hat{b}|, |\hat{c}| \geq 2$. We proceed as for Claim 3 and prove the following.

- (3) *If $\hat{a} = \hat{b}$ and $|\hat{a}| = |\hat{b}| = 2$. Then we can recolor $G \setminus \{v_1\}$ so that the hypothesis is not verified anymore.*

Proof. Let $\hat{a} = \hat{b} = \{\alpha, \beta\}$. Let γ be the color of e and δ the color of f . Note that $\gamma \in L(a)$ and $\delta \in L(b)$ since $|\hat{a}| = |\hat{b}| = 2$. Note also that α, β, γ and δ are all different.

We claim that any recoloring of $\{e, f, g_1, g_2, g_3, g_4\}$ such that the color of at least one of e, f has been changed breaks the hypothesis of (3). Indeed, assume w.l.o.g. that the color of e can be changed while recoloring only edges of $\{e, f, g_1, g_2, g_3, g_4\}$, and consider such a coloring. We have $\gamma \in \hat{a}$ since $\gamma \in L(a)$ and the only edge of $\{e, f, g_1, g_2, g_3, g_4\}$ that is incident to a is e , which is not colored in γ anymore. We have $\gamma \notin \hat{b}$ since $\gamma \notin \{\alpha, \beta, \delta\}$ and the only edge of $\{e, f, g_1, g_2, g_3, g_4\}$ that is incident to b is f , which was colored in δ . Thus $\hat{a} \neq \hat{b}$.

We prove that there exists such a recoloring. Aside from the constraints derived from $\{e, f, g_1, g_2, g_3, g_4\}$, both e and f have at most 7 constraints, edge g_1 has at most 4 constraints, edges g_2 and g_3 have at most 5 constraints, and g_4 has at most 7 constraints. Let L' be the list assignment of the colors available for those edges, when ignoring the constraints derived from $\{e, f, g_1, g_2, g_3, g_4\}$. Note that $|L'(e)|, |L'(f)|, |L'(g_4)| \geq 2$, $|L'(g_2)|, |L'(g_3)| \geq 4$ and $|L'(g_1)| \geq 5$. Let us build the directed graph D whose vertex set is $V(D) = \{e, f, g_1, g_2, g_3, g_4\}$ and where there is an edge from u to v if the color of u belongs to $L'(v)$. Let D_1 be the graph obtained from D by removing any vertex v such that there is no directed path from v to e . Let D_2 be the graph obtained from D by removing any vertex v such that there is no directed path from v to f . If $e \in D_2$ and $f \in D_1$, then there is a directed path from e to f and a directed path from f to e . So there exists a directed cycle that contains e , which we recolor accordingly. So we can assume that $e \notin D_2$ or $f \notin D_1$. We consider w.l.o.g. the case $f \notin D_1$. We consider four cases depending on the structure of D_1 .

- $V(D_1) = \{e\}$. Then we recolor e without conflict.
- $|V(D_1)| \geq 2$, and some vertex $v \neq e$ has in-degree at most $L'(v) - 2$. Then we recolor v , and recolor accordingly the path from v to e .
- $|V(D_1)| \geq 2$, and there is an edge from e to a vertex v . Then by definition of D_1 , there is a directed cycle that contains e , which we recolor accordingly.
- $|V(D_1)| \geq 2$, every vertex $v \neq e$ has in-degree at least $L'(v) - 1$, and e has out-degree 0. Since $f \notin D_1$, we have $\{g_1, g_2, g_3, g_4\} \cap D_1 \neq \emptyset$. Let j be the minimum i such that $g_i \in D_1$. Vertex g_j has in-degree at least $L'(g_j) - 1 \geq |V(D) \setminus \{f, g_1, \dots, g_j\}|$, and there is no edge from e to g_j , a contradiction.

◇

By (3), we can assume that we have a coloring of $G \setminus \{v_1\}$ that does not verify the hypothesis of (3). We apply Lemma 3 to the edges incident to v_1 . □

Claim 7. G cannot contain (C_7) .

Proof. By Claim 1, no two v_i are adjacent, nor is v_1 adjacent to a vertex of degree at most 7. Since every v_i is a weak neighbor of u , and $d(u) = 8$, the neighborhood of u forms a cycle (see Figure 15). We consider two cases depending on whether there is a vertex x such that v_2, x and v_3 appear consecutively around u .

- There is a vertex x such that v_3, x and v_4 appear consecutively around u .

W.l.o.g. the neighbors of u are, clockwise, $v_1, x_1, v_2, x_2, v_3, x_3, v_4, x_4$. Since v_2 is an E_2 -neighbor of u and $d(x_1) = 8$, we have $d(x_2) = 6$ and there is a vertex y of degree 6 such that (x_2, v_2, y) is a face. We name the edges according to Figure 15a. By minimality, we color $G \setminus \{a, \dots, s\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{n}| = |\hat{o}| = |\hat{p}| = |\hat{q}| = 2, |\hat{s}| = 3, |\hat{b}| = |\hat{f}| = |\hat{h}| = |\hat{i}| = |\hat{j}| = |\hat{k}| = 4, |\hat{m}| = |\hat{r}| = 5, |\hat{d}| = |\hat{e}| = |\hat{g}| = |\hat{l}| = 7$, and $|\hat{a}| = |\hat{c}| = 9$. Note that the edges k and s are not incident. Since $|\hat{k}| + |\hat{s}| > |\hat{r}|$, there exists $\alpha \in (\hat{k} \cap \hat{s}) \cup ((\hat{k} \cup \hat{s}) \setminus \hat{r})$. We color k and s in α if possible, in an arbitrary color otherwise. We can delete successively r, l and m . We color q . Note that the edges p and j are not incident. Since $|\hat{p}| + |\hat{j}| > |\hat{i}|$, there exists $\beta \in (\hat{p} \cap \hat{j}) \cup ((\hat{p} \cup \hat{j}) \setminus \hat{i})$. Thus we color p and j in β if possible, in an arbitrary color otherwise. We delete successively $i, a, c, e, g, d, h, b, f, n$ and o .

- There is no vertex x such that v_2, x and v_3 appear consecutively around u .

W.l.o.g. the neighbors of u are, clockwise, $v_1, x_1, v_3, x_2, v_2, x_3, v_4, x_4$, with $d(x_2) \geq d(x_3)$. We consider two cases depending on whether $d(x_2) = 6$.

- $d(x_2) = 6$.

W.l.o.g., since v_2 is an E_2 -vertex, there is a vertex y of degree 6 or 7 such that (y, v_2, x_3) is a face. We name the edges according to Figure 15b. By minimality, we color $G \setminus \{a, \dots, s\}$. W.l.o.g., we consider the worst case, i.e. $|\hat{k}| = |\hat{p}| = |\hat{q}| = |\hat{s}| = 2, |\hat{b}| = |\hat{h}| = |\hat{i}| = |\hat{j}| = |\hat{l}| = |\hat{r}| = 4, |\hat{o}| = 5, |\hat{d}| = |\hat{m}| = 6, |\hat{c}| = |\hat{f}| = |\hat{g}| = |\hat{n}| = 7$, and $|\hat{a}| = |\hat{e}| = 9$. Note that the edges r and l cannot be incident. Since $|\hat{r}| + |\hat{l}| > |\hat{m}|$, there exists $\alpha \in (\hat{r} \cap \hat{l}) \cup ((\hat{r} \cup \hat{l}) \setminus \hat{m})$. We color r and l in α if possible, in an arbitrary color otherwise. We can delete successively m, n and o . We color q, s and k successively. Note that p and j cannot be incident. Since $|\hat{p}| + |\hat{j}| > |\hat{i}|$, there exists $\beta \in (\hat{p} \cap \hat{j}) \cup ((\hat{p} \cup \hat{j}) \setminus \hat{i})$. Thus we color p and j in β if possible, in an arbitrary color otherwise. We delete successively i, a, e, g, f, c, d, h and b .

- $d(x_2) \geq 7$.

Since v_2 is an E_2 -vertex, we have $d(x_3) = 6$ and there is a vertex y of degree 6 such that (y, v_2, x_3) is a triangle. We name the edges according to Figure 15c. By minimality, we color $G \setminus \{a, \dots, s\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{k}| = |\hat{l}| = |\hat{p}| = |\hat{q}| = 2, |\hat{s}| = 3, |\hat{b}| = |\hat{d}| = |\hat{h}| = |\hat{i}| = |\hat{j}| = |\hat{m}| = 4, |\hat{o}| = |\hat{r}| = 5, |\hat{c}| = |\hat{f}| = |\hat{g}| = |\hat{n}| = 7$, and $|\hat{a}| = |\hat{e}| = 9$. Note that since G is a simple graph, the edges s and m cannot be incident. Since $|\hat{s}| + |\hat{m}| > |\hat{r}|$, there exists $\alpha \in (\hat{s} \cap \hat{m}) \cup ((\hat{s} \cup \hat{m}) \setminus \hat{r})$. We color m and s in α if possible, in an arbitrary color otherwise. We can delete successively r, n and o . We color successively q, l and k . Note that p and j cannot be incident. Since $|\hat{p}| + |\hat{j}| > |\hat{i}|$, there exists $\beta \in (\hat{p} \cap \hat{j}) \cup ((\hat{p} \cup \hat{j}) \setminus \hat{i})$. Thus we color p and j in β if possible, in an arbitrary color otherwise. We delete successively i, a, e, f, g, c, h, b and d .

□

Claim 8. G cannot contain (C_8) .

Proof. Note that since G is simple and $d(y) \neq d(v)$, all the vertices named here are distinct. We name the edges according to Figure 16. By minimality, we color $G \setminus \{a, \dots, f\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{a}| = |\hat{d}| = |\hat{f}| = 2, |\hat{c}| = |\hat{e}| = 3$ and $|\hat{b}| = 4$.

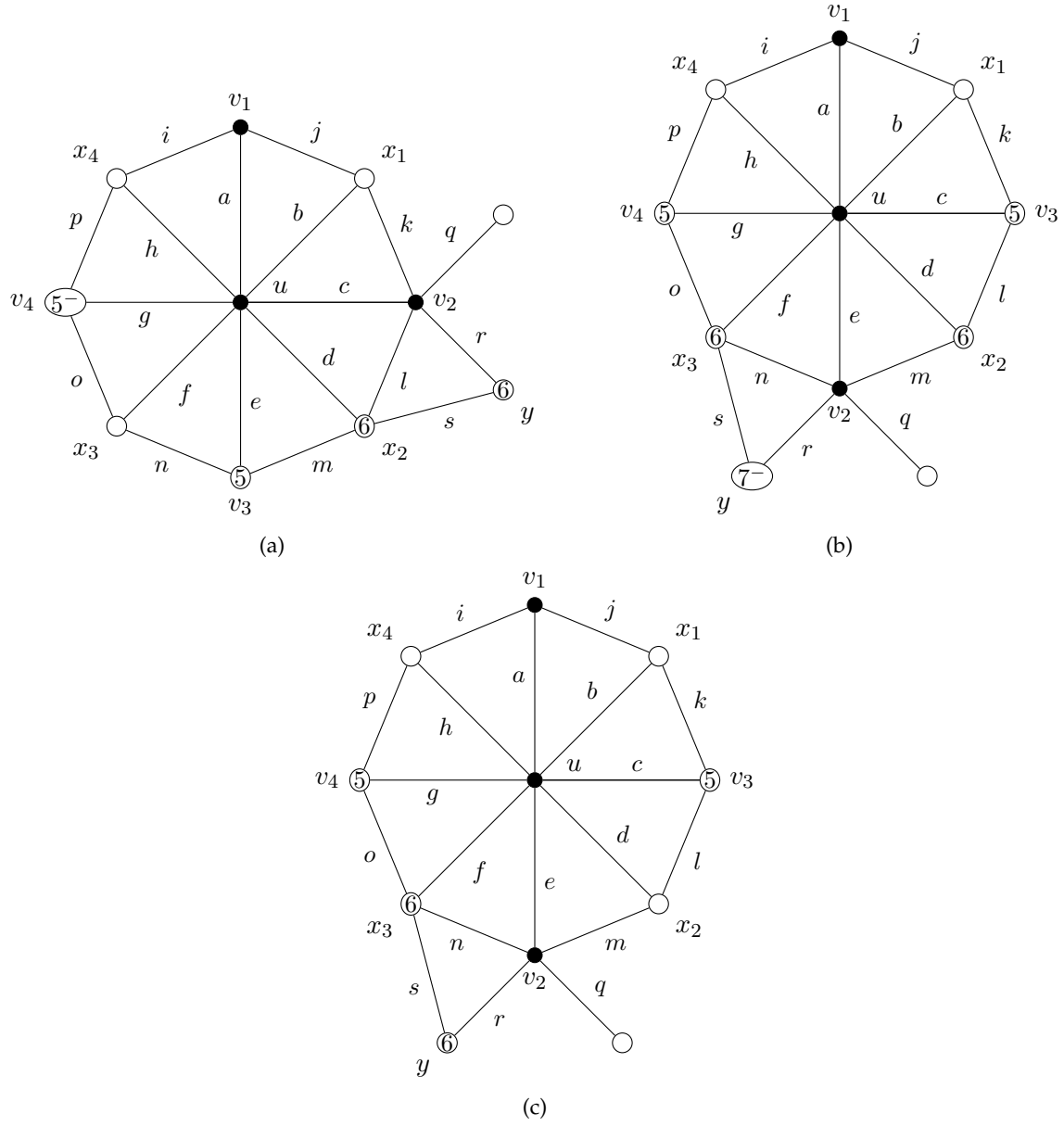


Figure 15: Notations of Claim 7

We consider two cases depending on whether $\hat{f} = \hat{d}$.

- $\hat{f} \neq \hat{d}$. We color f in a color that does not belong to \hat{d} . We apply Lemma 2 on (a, b, c, d, e) .
- $\hat{f} = \hat{d}$. We color e and c in a color that does not belong to \hat{f} . We color successively a, b, f and d .

□

Claim 9. G cannot contain (C_9) .

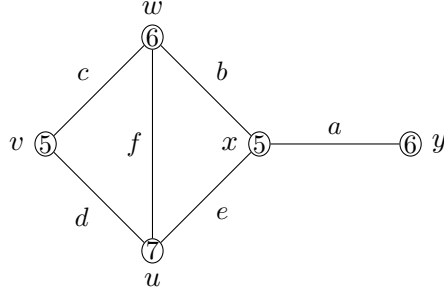


Figure 16: Notations of Claim 8

Proof. Note that by Claim 1, $\{v_1, v_2, v_3\}$ forms a stable set. We consider two cases depending on whether there are two weak neighbors w_1 and w_2 of u with $d(w_1) = d(w_2) = 4$ and a vertex x , such that (w_1, x, u) and (w_2, x, u) are faces.

- There are two weak neighbors w_1 and w_2 of u with $d(w_1) = d(w_2) = 4$ and a vertex x , such that (w_1, x, u) and (w_2, x, u) are faces.

We assume w.l.o.g. that the neighborhood of u is, clockwise, $y_1, v_1, x, v_2, y_2, v_3$ and z . We are in one of the following three cases: either $d(z) \leq 7$, or $d(y_2) \leq 7$, or $d(z) = d(y_2) = 8$.

- $d(z) \leq 7$.

We name the edges according to Figure 17a. By minimality, we color $G \setminus \{a, \dots, o\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{i}| = |\hat{l}| = |\hat{n}| = |\hat{o}| = 2$, $|\hat{a}| = |\hat{h}| = 3$, $|\hat{c}| = |\hat{e}| = |\hat{g}| = |\hat{j}| = |\hat{k}| = |\hat{m}| = 4$, $|\hat{f}| = 7$ and $|\hat{b}| = |\hat{d}| = 9$. We first prove the following.

- (4) We can color $a, c, e, f, g, h, i, j, k, l, m, n$ and o in such a way that $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$.

Proof. Let $B = \hat{b}$ and $D = \hat{d}$. If $\hat{b} = \hat{d}$, then we consider $\alpha \in \hat{i}$, and color i in α . If $\hat{b} \neq \hat{d}$, then we consider $\alpha \in \hat{d} \setminus \hat{b}$, and color i arbitrarily. We remove color α from \hat{k}, \hat{l} and \hat{m} . We color l . We consider two cases depending on whether $\hat{m} = \hat{n}$.

- * $\hat{n} \neq \hat{m}$.

Then we color m in a color that does not belong to \hat{n} . We color k . Since $|\hat{h}| + |\hat{c}| > |\hat{a}|$, there exists $\beta \in (\hat{h} \cap \hat{c}) \cup ((\hat{h} \cup \hat{c}) \setminus \hat{a})$. We color h and c in β if possible. We color successively j , and h or c if not colored already. We apply Lemma 2 on (a, g, o, n, e) . We color f .

- * $\hat{n} = \hat{m}$.

Since $|\hat{a}| + |\hat{o}| > |\hat{g}|$, there exists $\beta \in (\hat{a} \cap \hat{o}) \cup ((\hat{a} \cup \hat{o}) \setminus \hat{g})$. If $\beta \in \hat{e} \setminus \hat{m}$ or $\beta \notin \hat{a}$, we color e and o in β if possible, in an arbitrary color otherwise ($\notin \hat{m}$ in the case of e). We color n, m and k . We delete g , and we apply Lemma 1 on (h, j, c, a) . If $\beta \notin \hat{e} \setminus \hat{m}$ and $\beta \in \hat{a}$, we color a and o in β if possible, in an arbitrary color otherwise. Note that a is colored in β , and that β does not belong to \hat{e} or belongs to \hat{m} , in which case one of $\{m, n\}$ will be colored in β . We color successively n, m, k, h, j, c , and e . We color g , and f .

Assume $|\hat{b}| = |\hat{d}| = 1$. Then the colors of the edges incident to b are all different and belong to B . We consider two cases depending on whether $B = D$.

- * $B = D$. Since i is colored in α , no edge in $\{a, c, e, g\}$ is colored in α , and $\alpha \in D$. By construction, none of $\{k, l, m\}$ is colored in α . Thus $\alpha \in \hat{d}$ and $\alpha \notin \hat{b}$, so $\hat{b} \neq \hat{d}$.
- * $B \neq D$. Since $\alpha \notin B$, no edge in $\{a, c, e, g\}$ is colored in α . By construction, none of $\{k, l, m\}$ is colored in α . Thus $\alpha \in \hat{d}$ and $\alpha \notin \hat{b}$, so $\hat{b} \neq \hat{d}$.

◇

By (4), we color $a, c, e, f, g, h, i, j, k, l, m, n$ and o in such a way that $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$. We color b and d .

– $d(y_2) \leq 7$.

We name the edges according to Figure 17b. By minimality, we color $G \setminus \{a, \dots, m\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{g}| = |\hat{i}| = |\hat{l}| = 2$, $|\hat{a}| = |\hat{h}| = 3$, $|\hat{c}| = |\hat{e}| = |\hat{j}| = |\hat{k}| = |\hat{m}| = 4$, $|\hat{f}| = 5$ and $|\hat{b}| = |\hat{d}| = 9$. We first prove the following.

(5) *We can color $a, c, e, f, g, h, i, j, k, l$ and m in such a way that, afterwards, $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$.*

Proof. If $\hat{b} = \hat{d}$, then we consider $\alpha \in \hat{l}$, and color l in α . If $\hat{b} \neq \hat{d}$, then we consider $\alpha \in \hat{b} \setminus \hat{d}$, and color l arbitrarily. We remove color α from \hat{h}, \hat{i} and \hat{j} . We color successively $i, h, j, a, g, c, k, e, m$ and f . By the same analysis as in the previous case, $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$. ◇

By (5), we color $a, c, e, f, g, h, i, j, k, l$ and m in such a way that $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$. We color b and d .

– $d(z) = d(y_2) = 8$.

Then either v_3 is a weak neighbor of u of degree 4, or v_3 is a weak neighbor of u of degree 5 adjacent to a vertex of degree 6. We will deal with the two cases at once. We consider that v_3 is of degree 5 in both cases, by adding a neighbor of degree 6 to v_3 if it is of degree 4: a proper coloring of this graph will yield a proper coloring of the initial graph. We name the edges according to Figure 17b.

By minimality, we color $G \setminus \{a, \dots, q\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{i}| = |\hat{l}| = |\hat{p}| = 2$, $|\hat{a}| = |\hat{g}| = |\hat{h}| = |\hat{o}| = 3$, $|\hat{c}| = |\hat{e}| = |\hat{j}| = |\hat{k}| = |\hat{m}| = |\hat{n}| = |\hat{q}| = 4$ and $|\hat{b}| = |\hat{d}| = |\hat{f}| = 9$. We first prove the following.

(6) *We can color $a, c, e, f, g, h, i, j, k, l, m, n, o, p$ and q in such a way that, afterwards, $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$.*

Proof. If $\hat{b} = \hat{d}$, then we consider $\alpha \in \hat{l}$, and color i in α . If $\hat{b} \neq \hat{d}$, then we consider $\alpha \in \hat{d} \setminus \hat{b}$, and color i arbitrarily. We remove color α from \hat{k}, \hat{l} and \hat{m} .

We color l . Since $|\hat{g}| + |\hat{p}| > |\hat{o}|$, there exists $\beta \in (\hat{g} \cap \hat{p}) \cup ((\hat{g} \cup \hat{p}) \setminus \hat{o})$. We color g and p in β if possible, in an arbitrary color otherwise. We color m so that $\hat{e} \neq \hat{a}$ if $|\hat{a}| = |\hat{e}| = 2$, which is possible as $|\hat{m}| \geq 2$. We color k , and we apply Lemma 2 on (e, a, h, j, c) . We color n, o, q and f .

By the same analysis as in the two previous cases, we have $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$. ◇

By (6), we color $a, c, e, f, g, h, i, j, k, l, m, n, o, p$ and q in such a way that $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$. We color b and d .

- There are no two weak neighbors w_1 and w_2 of u with $d(w_1) = d(w_2) = 4$ for which there exists a vertex x such that (w_1, x, u) and (w_2, x, u) are faces.

Then v_3 must be a vertex of degree 5. W.l.o.g., the neighborhood of u is, clockwise, $y_1, v_1, y_2, v_3, y_3, v_2, y_4$. We consider two cases depending on whether $d(y_2) = d(y_3) = 8$.

- $d(y_2) \leq 7$ or $d(y_3) \leq 7$.

Consider w.l.o.g. that $d(y_2) \leq 7$. We name the edges according to Figure 17d. By minimality, we color $G \setminus \{a, \dots, o\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{i}| = |\hat{l}| = |\hat{n}| = 2$, $|\hat{a}| = |\hat{g}| = |\hat{h}| = |\hat{k}| = |\hat{o}| = 3$, $|\hat{e}| = |\hat{m}| = 4$, $|\hat{c}| = |\hat{j}| = 5$, $|\hat{d}| = 7$ and $|\hat{b}| = |\hat{f}| = 9$.

Note that the edges k and h are not incident. Since $|\hat{k}| + |\hat{h}| > |\hat{j}|$, there exists $\alpha \in (\hat{k} \cap \hat{h}) \cup ((\hat{k} \cup \hat{h}) \setminus \hat{j})$. We color k and h in α if possible, in an arbitrary color otherwise. We can delete successively j, b, i, f, d and c . We color a, l and n . We apply Lemma 1 on (e, m, o, g) .

- $d(y_2) = d(y_3) = 8$.

Then v_3 must be a weak neighbor of degree 5 whose two other neighbors are of degree 6 and 7, respectively. We name the edges according to Figure 17e. By minimality, we color $G \setminus \{a, \dots, q\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{i}| = |\hat{n}| = 2$, $|\hat{a}| = |\hat{g}| = |\hat{h}| = |\hat{o}| = |\hat{q}| = 3$, $|\hat{c}| = |\hat{e}| = |\hat{j}| = |\hat{k}| = |\hat{l}| = |\hat{m}| = |\hat{p}| = 4$, and $|\hat{b}| = |\hat{d}| = |\hat{f}| = 9$. We first prove the following.

(7) We can color $a, c, e, f, g, h, i, j, k, l, m, n, o, p$ and q in such a way that, afterwards, $\hat{b} \neq \hat{f}$ if $|\hat{b}| = |\hat{f}| = 1$.

Proof. If $\hat{b} = \hat{f}$, then we consider $\alpha \in \hat{n}$, and color n in α . If $\hat{b} \neq \hat{f}$, then we consider $\alpha \in \hat{b} \setminus \hat{d}$, and color n arbitrarily. We remove color α from \hat{h}, \hat{i} and \hat{j} . We color successively $i, h, j, k, c, a, g, e, o, m, l, q, p$ and d . By the same analysis as in the previous cases, we have $\hat{b} \neq \hat{f}$ if $|\hat{b}| = |\hat{f}| = 1$. \diamond

By (7), we color $a, c, e, f, g, h, i, j, k, l, m, n, o, p$ and q in such a way that $\hat{b} \neq \hat{f}$ if $|\hat{b}| = |\hat{f}| = 1$. We color b and f . □

Claim 10. G cannot contain (C_{10}) .

Proof. We consider two cases depending on whether v_2 and u have a common neighbor of degree 6.

- Vertices v_2 and u have a common neighbor y of degree 6.

By definition of an S_3 -neighbor, vertex v_2 has two other neighbors of degree 7 and 6, respectively. We name the edges according to Figure 18a. Since the graph is simple, there is no $1 \leq i \leq 3$ such that the edges e and c_i are incident.

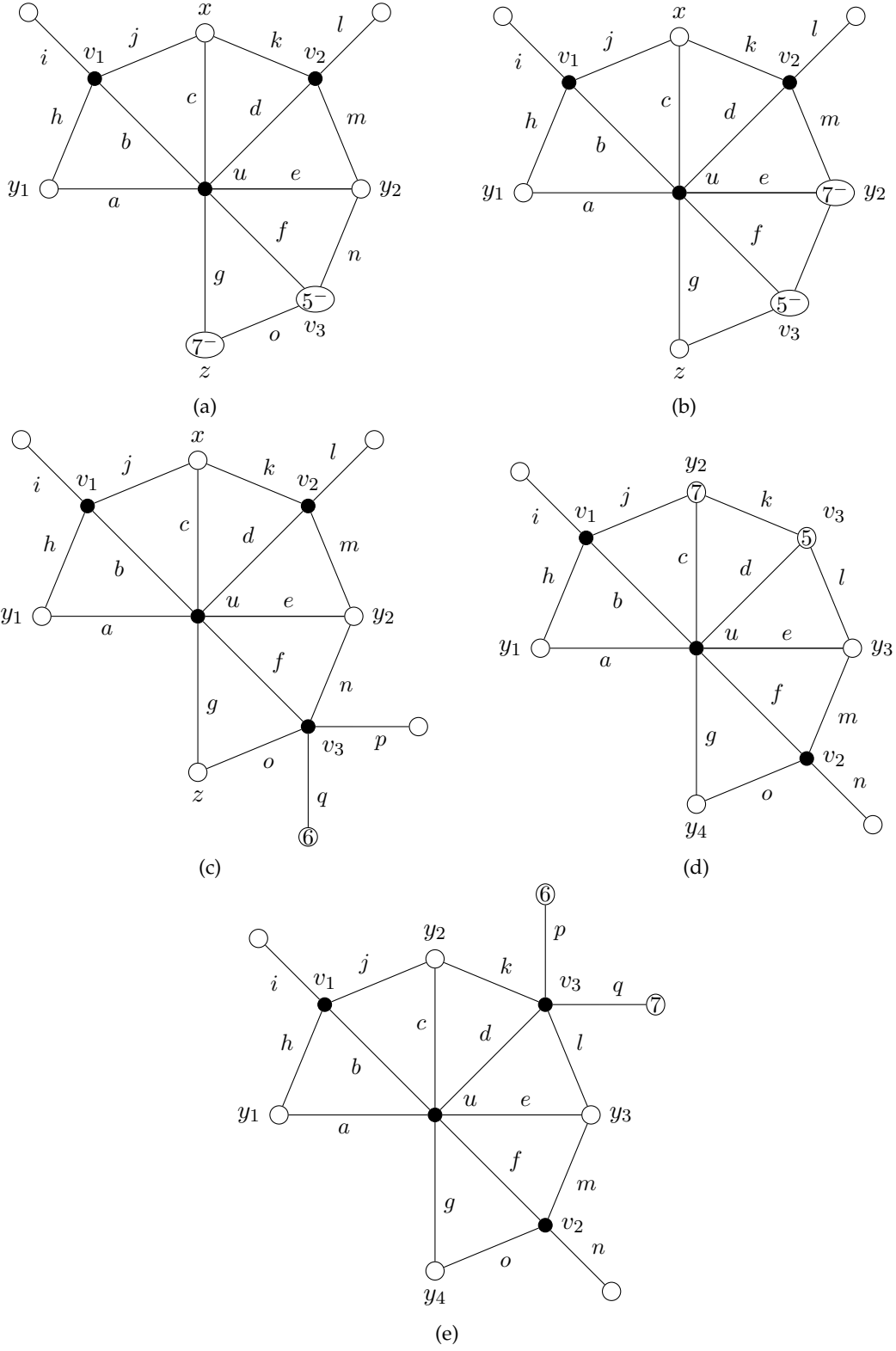


Figure 17: Notations of Claim 9

By minimality, we color $G \setminus \{a, b_1, b_2, c_1, c_2, c_3, d, e\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{b}_1| = |\hat{c}_1| = |\hat{e}| = 2$, $|\hat{b}_2| = |\hat{c}_2| = 3$, $|\hat{c}_3| = 4$, $|\hat{d}| = 5$, and $|\hat{a}| = 6$.

Since $|\hat{e}| + |\hat{c}_3| > |\hat{d}|$, there exists $\alpha \in (\hat{e} \cap \hat{c}_3) \cup ((\hat{e} \cup \hat{c}_3) \setminus \hat{d})$. If $\alpha \in \hat{c}_3$, let i be the minimum integer such that $\alpha \in \hat{c}_i$. If $\alpha \notin \hat{c}_3$, then $\alpha \in \hat{e} \setminus (\hat{d} \cup \hat{c}_3)$, let i be 1. We color e and c_i in α if possible, in an arbitrary color otherwise (by choice of i , if c_i is not colored in α then $i = 1$). We delete d . If $i \neq 3$, edge c_3 is not colored and we delete it. Then, if $i \neq 2$, edge c_2 is not colored, and either $i = 3$ and c_3 is colored in α (which was not an available color for c_2 by choice of i), or $i = 1$ and c_3 has been deleted; In both cases, we can delete c_2 . Then, if $i \neq 1$, edge c_1 is not colored, and the edges c_2 and c_3 are deleted or colored in α (which was not an available color for c_1 by choice of i), so we can delete c_1 . We delete successively a, b_2, b_1 .

- Vertices v_2 and u have no common neighbor of degree 6.

Then, by definition of an S_3 -vertex, the neighborhood of v_2 is, clockwise, (u, y_1, z_1, z_2, y_2) , with $d(y_1) = d(y_2) = 7$ and $d(z_1) = d(z_2) = 6$. We name the edges according to Figure 18b. By minimality, we color $G \setminus \{a, \dots, k\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{f}| = |\hat{g}| = |\hat{h}| = |\hat{j}| = 2$, $|\hat{i}| = |\hat{k}| = 3$, $|\hat{b}| = |\hat{e}| = 5$, and $|\hat{a}| = |\hat{c}| = |\hat{d}| = 6$. We first prove the following.

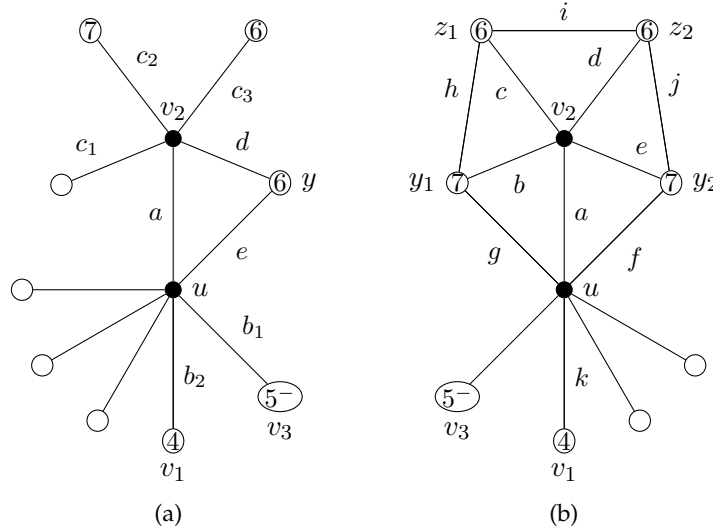


Figure 18: Notations of Claim 10

- (8) We can color f, g, h, i, j and k in such a way that, afterwards, $\hat{c} \neq \hat{d}$ if $|\hat{c}| = |\hat{d}| = 4$.

Proof. We consider two cases depending on whether $\hat{c} = \hat{d}$.

- $\hat{c} = \hat{d}$. We apply Lemma 1 on (f, g, h, j) by considering that f and j are incident so they receive different colors. We color i and k . The new constraints of c are i and h , and the new constraints of d are i and j . Since h and j receive distinct colors, we have $|\hat{c}| \geq 5$ or $|\hat{d}| \geq 5$ or $\hat{c} \neq \hat{d}$.
- $\hat{c} \neq \hat{d}$. Let $\alpha \in \hat{c}, \alpha \notin \hat{d}$. We color h in a color other than α . We color g, f, j, i and k successively. Thus, either $|\hat{d}| \geq 5$ or $\alpha \in \hat{c}$ so $\hat{c} \neq \hat{d}$.

◇

By (8), we color f, g, h, i, j and k in such a way that $\hat{c} \neq \hat{d}$ if $|\hat{c}| = |\hat{d}| = 4$. We color a, b and e . Either $|\hat{c}| \geq 2$ (resp. $|\hat{d}| \geq 2$), and we color d and c (resp. c and d). Or $|\hat{c}| = |\hat{d}| = 1$ and $\hat{c} \neq \hat{d}$, we color d and c independently.

□

Claim 11. G cannot contain (C_{11}) .

Proof. We name the edges according to Figure 19. By minimality, we color $G \setminus \{a, \dots, e\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{d}| = |\hat{e}| = 2$, $|\hat{a}| = |\hat{c}| = 3$ and $|\hat{b}| = 4$. We consider two cases depending on whether $\hat{e} \subset \hat{b}$.

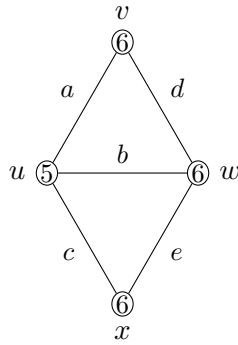


Figure 19: Notations of Claim 11

- $\hat{e} \not\subset \hat{b}$. Then we color e in a color that does not belong to \hat{b} . We can delete successively b, a, c and d .
- $\hat{e} \subset \hat{b}$. Then, since $|\hat{c}| + |\hat{d}| > |\hat{b}|$ and $\hat{d} \subset \hat{b}$, there exists $\alpha \in (\hat{c} \cap \hat{d}) \cup ((\hat{c} \cup \hat{d}) \setminus \hat{b})$. We color c and d in α if possible, in an arbitrary color otherwise. Note that since $\hat{e} \subset \hat{b}$, we have $|\hat{e}| \geq 1$ in both cases. We delete successively b, e and a .

□

Lemma 4 holds by Claims 1 to 11.

□

5 Discharging rules

We design discharging rules R_1, R_2, \dots, R_{11} (see Figure 20):

For any face f of degree at least 4,

- Rule R_1 is when $d(f) = 4$ and f is incident to a vertex v of degree $d(v) \leq 5$. Then f gives 1 to v .

- Rule R_2 is when $d(f) \geq 5$ and f is incident to a vertex v of degree $d(v) \leq 5$. Then f gives 2 to v .

For any vertex u of degree at least 7,

- Rule R_3 is when u has a weak neighbor v of degree 3. Then u gives 1 to v .
- Rule R_4 is when u has a semi-weak neighbor v of degree 3. Then u gives $\frac{1}{2}$ to v .
- Rule R_5 is when u has a weak neighbor v of degree 4. Then u gives $\frac{1}{2}$ to v .

For any vertex u of degree 8,

- Rule R_6 is when u has an E_2 -neighbor v . Then u gives $\frac{1}{2}$ to v .
- Rule R_7 is when u has an E_3 -neighbor v . Then u gives $\frac{1}{3}$ to v .
- Rule R_8 is when u has an E_4 -neighbor v . Then u gives $\frac{1}{4}$ to v .

For any vertex u of degree 7,

- Rule R_9 is when u has an S_2 -neighbor v . Then u gives $\frac{1}{2}$ to v .
- Rule R_{10} is when u has an S_3 -neighbor v . Then u gives $\frac{1}{3}$ to v .
- Rule R_{11} is when u has an S_4 -neighbor v . Then u gives $\frac{1}{4}$ to v .

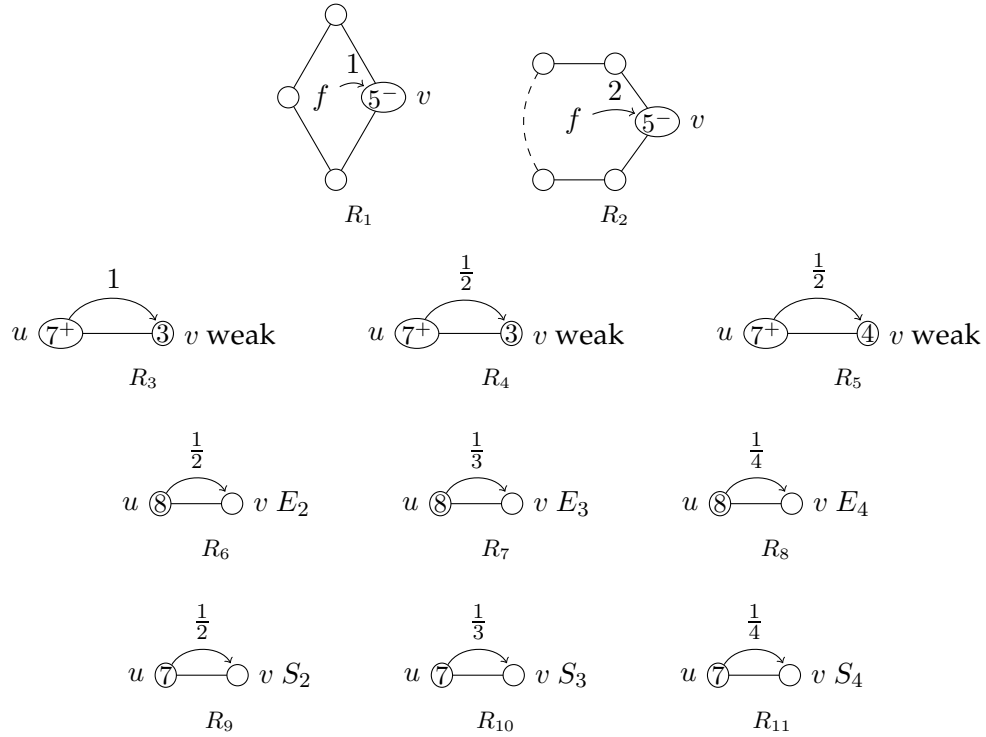


Figure 20: Discharging rules.

Note that according to these rules, only vertices of degree at most 5 receive weight, and only faces of degree at least 4 and vertices of degree at least 7 give weight. Note that the notation E_i and S_i corresponds to the fact that a vertex u gives a weight of $\frac{1}{i}$ to every E_i - or S_i -neighbor.

Lemma 5. *A planar graph G with $\Delta(G) \leq 8$ that does not contain Configurations (C_1) to (C_{11}) is a stable set.*

Proof. We can assume without loss of generality that G is connected (if it is not, we simply consider a connected component of G , as it verifies the same hypothesis). Assume by contradiction that G is not a single vertex. Thus G is connected and contains at least one edge. According to Configuration (C_1) , every vertex x of G verifies $d(x) \geq 3$. We consider a planar embedding of G . We attribute to each vertex u a weight of $d(u) - 6$, and to each face a weight of $2d(f) - 6$. We apply discharging rules R_1, R_2, \dots, R_{11} . We show that all the faces and vertices have a weight of at least 0 in the end.

Note that the degree of a face is the number of vertices on its boundary, while walking through a facial walk (i.e. some vertices are counted with multiplicity). The discharging rules on the faces also apply with multiplicity: R_1 and R_2 apply to each vertex of degree at most 5 incident to f as many times as it appears on the boundary of f .

Let f be a face in G . By Configuration (C_1) , no two vertices of degree at most 5 are adjacent. Thus f is incident to at most $\lfloor \frac{d(f)}{2} \rfloor$ vertices of degree ≤ 5 . We consider four cases depending on $d(f)$.

1. $d(f) = 3$. Then f has an initial weight of 0 and gives nothing, so it has a final weight of at least 0.
2. $d(f) = 4$. Face f is incident to at most 2 vertices of degree ≤ 5 . So f has an initial weight of 2 and gives at most two times 1 according to R_1 . Thus f has a final weight of at least $2 - 2 \times 1 \geq 0$.
3. $d(f) = 5$. Face f is incident to at most 2 vertices of degree ≤ 5 . So f has an initial weight of 4 and gives at most two times 2 according to R_2 . Thus f has a final weight of at least $4 - 2 \times 2 \geq 0$.
4. $d(f) \geq 6$. Face f is incident to at most $\lfloor \frac{d(f)}{2} \rfloor \leq \frac{d(f)}{2}$ vertices of degree ≤ 5 . So f has an initial weight of $2 \times d(f) - 6$ and gives at most $\frac{d(f)}{2}$ times 2 according to R_2 . Thus f has a final weight of at least $2 \times d(f) - 6 - 2 \times \frac{d(f)}{2} = d(f) - 6 \geq 0$.

So all the faces have a final weight of at least 0 after application of the discharging rules. Let us now prove that the same holds for the vertices.

Let x be a vertex of G . We consider different cases corresponding to the value of $d(x)$.

1. $d(x) = 3$. Vertex x has an initial weight of -3 . We show that it receives at least 3, thus has a non-negative final weight. By Configuration (C_1) , the three neighbors of x are of degree 8. We consider four cases depending on the degrees of the three faces f_1, f_2 and f_3 incident to x . We assume $d(f_1) \geq d(f_2) \geq d(f_3)$. Let u_1, u_2 and u_3 be the three neighbors of x , where for every $i \in \{1, 2, 3\}$, the edge (x, u_i) belongs to f_{i-1} and f_i (subscripts taken modulo 3).

- (a) $d(f_1) \geq 5$ and $d(f_2) \geq 4$.
So x receives 2 from f_1 by R_2 , and at least 1 from f_2 by R_1 or R_2 .
- (b) $d(f_1) = d(f_2) = d(f_3) = 4$.
So x receives 1 from each f_i by R_1 .
- (c) $d(f_1) = d(f_2) = 4$ and $d(f_3) = 3$.
So x receives 1 from both f_1 and f_2 by R_1 . Besides, x is a semi-weak neighbor of u_1 and u_3 , so x receives $\frac{1}{2}$ from u_1 and u_2 by R_4 .
- (d) $d(f_1) \geq 5$ and $d(f_2) = d(f_3) = 3$.
So x receives 2 from f_1 by R_2 . Vertex x is a weak neighbor of u_3 , so x receives 1 from u_3 by R_3 .
- (e) $d(f_1) = 4$, and $d(f_2) = d(f_3) = 3$.
So x receives 1 from f_1 by R_1 . Besides, x is a weak neighbor of u_3 and a semi-weak neighbor of u_1 and u_2 , so x receives 1 from u_3 by R_3 , and $\frac{1}{2}$ from both u_1 and u_2 by R_4 .
- (f) $d(f_1) = d(f_2) = d(f_3) = 3$.
Then x is a weak neighbor of u_1, u_2 and u_3 , so x receives 1 from u_1, u_2 and u_3 by R_3 .
2. $d(x) = 4$. Vertex x has an initial weight of -2 . We show that it receives at least 2, thus has a non-negative final weight. By Configuration (C_1) , the four neighbors u_1, u_2, u_3 and u_4 of x are of degree at least 7. We consider three cases depending on how many triangles are incident to x .
- (a) *Vertex x is incident to at most 2 triangles.*
Then x is incident to at least two faces f_1 and f_2 with $d(f_1), d(f_2) \geq 4$. So x receives at least 1 from both f_1 and f_2 by R_1 or R_2 .
- (b) *Vertex x is incident to exactly 3 triangles $(x, u_1, u_2), (x, u_2, u_3)$ and (x, u_3, u_4) .*
Then x is incident to a face f_1 with $d(f_1) \geq 4$. So x receives at least 1 from f_1 by R_1 or R_2 . Besides, x is a weak neighbor of u_2 and u_3 , so x receives $\frac{1}{2}$ from both u_2 and u_3 by R_5 .
- (c) *Vertex x is incident to 4 triangles.*
Then x is a weak neighbor of u_1, u_2, u_3 and u_4 , so x receives $\frac{1}{2}$ from u_1, u_2, u_3 and u_4 by R_5 .
3. $d(x) = 5$. Vertex x has an initial weight of -1 . We show that it receives at least 1, thus has a non-negative final weight. By Configuration (C_1) , the five consecutive neighbors u_1, u_2, u_3, u_4 and u_5 of x are of degree at least 6.
- In the case where x is incident to a face f with $d(f) \geq 4$, vertex x receives at least 1 from f by R_1 or R_2 . So we can assume that x is incident to five triangles $(x, u_1, u_2), (x, u_2, u_3), (x, u_3, u_4), (x, u_4, u_5)$ and (x, u_5, u_1) . We consider four cases depending on the number of vertices of degree 6 incident to x .
- (a) *Vertex x has at least three neighbors of degree 6.*
By Configuration (C_{11}) , they cannot appear consecutively around x , so they are exactly three. Without loss of generality, we assume $d(u_1) = d(u_2) = d(u_4) = 6$, hence $d(u_3), d(u_5) \geq 7$. Then x is an E_2 - or S_2 -neighbor of u_3 and u_5 , so it receives $\frac{1}{2}$ from both u_3 and u_5 by R_6 or R_9 .

(b) *Vertex x has exactly two neighbors of degree 6.*

We consider two cases depending on whether these vertices of degree 6 appear consecutively around x .

i. *Vertex x has two consecutive neighbors of degree 6.*

We can assume w.l.o.g. that $d(u_1) = d(u_2) = 6$, and that $d(u_3) \geq d(u_5)$. We consider three cases depending on $d(u_3)$ and $d(u_5)$.

A. $d(u_3) = d(u_5) = 8$.

Then x is an E_2 -neighbor of u_3 and u_5 , so x receives $\frac{1}{2}$ from both u_3 and u_5 by R_6 .

B. $d(u_3) = 8, d(u_5) = 7$.

Then x is an E_2 -neighbor of u_3 , an S_3 - or S_4 -neighbor of u_5 (depending on the degree of u_4), and an S_4 - or E_3 -neighbor of u_4 , so x receives $\frac{1}{2}$ from u_3 by R_6 , and at least $\frac{1}{4}$ from both u_4 and u_5 by R_7, R_{10} or R_{11} .

C. $d(u_3) = d(u_5) = 7$.

Then x is an S_3 -neighbor of u_3 and u_5 , and an S_3 - or E_3 -neighbor of u_4 , so x receives $\frac{1}{3}$ from u_3, u_4 and u_5 by R_7 or R_{10} .

ii. *Vertex x has no two consecutive neighbors of degree 6.*

We can assume without loss of generality that $d(u_1) = d(u_4) = 6$ and that $d(u_2) \geq d(u_3)$. We consider two cases depending on $d(u_3)$.

A. $d(u_3) = 8$.

Then $d(u_2) = 8$. Vertex x is an E_3 - or S_2 -neighbor of u_5 , and an E_3 -neighbor of u_2 and u_3 , so x receives at least $\frac{1}{3}$ from u_2, u_3 and u_5 by R_7 or R_9 .

B. $d(u_3) = 7$.

Then x is an E_2 - or S_2 -neighbor of u_5 , and an S_3 -, S_4 - or E_3 -neighbor of u_2 and u_3 , so x receives $\frac{1}{2}$ from u_5 by R_6 or R_9 , and at least $\frac{1}{4}$ from u_2 and u_3 by R_7, R_{10} or R_{11} .

(c) *Vertex x has exactly one neighbor of degree 6.*

We can assume without loss of generality that $d(u_1) = 6$, and $d(u_2) \geq d(u_5)$ or $d(u_3) \geq d(u_4)$ if $d(u_2) = d(u_5)$. We consider three cases depending on $d(u_5)$ and $d(u_3)$.

i. $d(u_5) = 8$ and $d(u_3) = d(u_4)$.

Then x is an E_3 -neighbor of u_2 and u_5 , so it receives $\frac{1}{3}$ from both by R_7 . Besides, since $d(u_3) = d(u_4)$, vertex x is an S_4 - or E_4 -neighbor of u_3 and u_4 , so it receives $\frac{1}{4}$ from both by R_8 or R_{11} .

ii. $d(u_5) = 8$ and $d(u_3) \neq d(u_4)$.

Then $d(u_2) = d(u_3) = 8$ and $d(u_4) = 7$. Vertex x is an E_3 -neighbor of u_2, u_3 and u_5 , so it receives $\frac{1}{3}$ from each by R_7 .

iii. $d(u_5) = 7$.

Then vertex x is an E_3 -, E_4 - or S_4 -neighbor of every u_i for $i \in \{2, 3, 4, 5\}$, so it receives at least $\frac{1}{4}$ from each by R_7, R_8 or R_{11} .

(d) *Vertex x has no neighbor of degree 6.*

We consider three cases depending on the degrees of the u_i 's.

i. *Vertex x has at least 4 neighbors of degree 8.*

Then x is an E_3 - or E_4 -neighbor of each of them, so it receives at least $\frac{1}{4}$ from each by R_7 or R_8 .

- ii. *Vertex x has two consecutive neighbors of degree 7.*
 We consider w.l.o.g. that $d(u_1) = d(u_2) = 7$. Then x is an S_4 -neighbor of u_1 and u_2 , so it receives at least $\frac{1}{4}$ from each by R_{11} . Vertex x is also an S_4 - or E_3 -neighbor of u_3 and u_5 , so it receives at least $\frac{1}{4}$ from each by R_7 or R_{11} .
 - iii. *Vertex x has at most 3 neighbors of degree 8, and has no two consecutive neighbors of degree 7.*
 Since x is only adjacent to vertices of degree 7 or 8, we consider w.l.o.g. that $d(u_1) = d(u_3) = 7$, and $d(u_2) = d(u_4) = d(u_5) = 8$. Then x is an E_3 -neighbor of u_2, u_4 and u_5 , so it receives $\frac{1}{3}$ from each by R_7 .
4. $d(x) = 6$. Vertex x has an initial weight of 0, gives nothing away, and has a final weight of at least 0.
5. $d(x) = 7$. Vertex x has an initial weight of 1. We show that it gives at most 1, thus has a non-negative final weight. By Configuration (C_1) , the neighbors of x have degree at least 4, and x has at most 3 weak neighbors of degree at most 5. We consider four cases depending on the weak neighbors of x .
- (a) *Vertex x has an S_2 -neighbor v .*
 Let $v, w_1, w_2, w_3, w_4, w_5$ and w_6 be the consecutive neighbors of x . By definition of an S_2 -neighbor, $d(w_1) = d(w_6) = 6$. By Configuration (C_8) , if w_2 (resp. w_5) is a weak neighbor of x , then $d(w_2) > 5$ (resp. $d(w_5) > 5$). Assume w.l.o.g. that $d(w_3) \geq d(w_4)$. Then by Configuration (C_1) , if w_3 and w_4 are adjacent then $d(w_3) > 5$. Thus x has at most two weak neighbors of degree at most 5: v and possibly w_4 . Besides, $d(v), d(w_4) > 3$. By Rules R_5, R_9, R_{10} and R_{11} , vertex x gives at most $\frac{1}{2}$ to each.
 - (b) *Vertex x has at least two weak neighbors of degree 4.*
 By Configuration (C_9) , x is adjacent to no other weak neighbor of degree 4, and no S_2, S_3 or S_4 -neighbor. Thus x gives $\frac{1}{2}$ to each of the two weak neighbors of degree 4 by R_5 .
 - (c) *Vertex x has exactly one weak neighbor v of degree 4 and no S_2 -neighbor.*
 If x has an S_3 -neighbor v_2 , then by Configuration (C_{10}) , it has no other neighbor of degree at most 5. Thus x gives $\frac{1}{2}$ to v by R_5 , $\frac{1}{3}$ to v_2 by R_{10} .
 If x has no S_3 -neighbor, then x has at most two other weak neighbors v_1 and v_2 of degree at most 5, which are of degree 5 by assumption. So x gives $\frac{1}{2}$ to v by R_5 , $\frac{1}{4}$ to v_1 and v_2 by R_{11} .
 - (d) *Vertex x has no weak neighbor of degree 4, and no S_2 -neighbor.*
 Vertex x has at most three weak neighbors v_1, v_2 and v_3 of degree at most 5, which are of degree 5 by assumption. So x gives at most $\frac{1}{3}$ to each by R_{10} or R_{11} .
6. $d(x) = 8$. Vertex x has an initial weight of 2. We show that it gives at most 2, thus has a non-negative final weight. By Configurations (C_1) and (C_2) , vertex x has at most 4 neighbors that are either semi-weak with degree 3 or weak with degree at most 5. We consider eight cases depending on the neighborhood of x .
- (a) *Vertex x has at least two weak neighbors v_1 and v_2 of degree 3.*
 Then by Configuration (C_3) , vertex x has exactly two neighbors of degree at most 5. Thus x gives 1 to v_1 and v_2 by R_3 .

- (b) *Vertex x has exactly one weak neighbor v_1 of degree 3, and at least one semi-weak neighbor v_2 of degree 3.*
 Then by Configuration (C_4) , vertex x has at most one other neighbor v_3 of degree at most 5. By assumption, vertex v_3 is not a weak neighbor of x of degree 3, so x gives at most $\frac{1}{2}$ to v_3 by R_4, R_5, R_6, R_7 or R_8 . Vertex x gives 1 to v_1 by R_3 , and $\frac{1}{2}$ to v_2 by R_4 .
- (c) *Vertex x has exactly one weak neighbor v_1 of degree 3, no semi-weak neighbor of degree 3, and at least two weak neighbors v_2 and v_3 of degree 4.*
 Then, by Configuration (C_5) , vertex x has no other weak neighbor of degree at most 5. By assumption, it has no semi-weak neighbor of degree 3. So x gives 1 to v_1 by R_3 , $\frac{1}{2}$ to v_2 and v_3 by R_5 .
- (d) *Vertex x has exactly one weak neighbor v_1 of degree 3, no semi-weak neighbor of degree 3, exactly one weak neighbor v_2 of degree 4, and at least one E_2 - or E_3 -neighbor v_3 .*
 By definition of E_2 - and E_3 -neighbor, vertices x and v_3 have a common neighbor v_4 of degree at most 7, which by Configuration (C_1) has degree 6 or 7. Then, by Configuration (C_6) , vertex x has no other neighbor of degree at most 5. So x gives 1 to v_1 by R_3 , $\frac{1}{2}$ to v_2 by R_5 , at most $\frac{1}{2}$ to v_3 by R_6 or R_7 .
- (e) *Vertex x has exactly one weak neighbor v_1 of degree 3, no semi-weak neighbor of degree 3, exactly one weak neighbor v_2 of degree 4, and no E_2 - or E_3 -neighbor.*
 Then x has at most two other weak neighbors v_3 and v_4 of degree at most 5, which are by assumption E_4 -neighbors. So x gives 1 to v_1 by R_3 , $\frac{1}{2}$ to v_2 by R_5 , $\frac{1}{4}$ to v_3 and v_4 by R_8 .
- (f) *Vertex x has exactly one weak neighbor v_1 of degree 3, no semi-weak neighbor of degree 3, no weak neighbor v_2 of degree 4, and at least an E_2 -neighbor v_2 .*
 Then by Configuration (C_7) , vertex x has at most one other weak neighbor v_3 of degree at most 5, which is by assumption of degree 5. So x gives 1 to v_1 by R_3 , at most $\frac{1}{2}$ to v_2 and v_3 by R_6, R_7 or R_8 .
- (g) *Vertex x has exactly one weak neighbor v_1 of degree 3, no weak neighbor v_2 of degree 4, no semi-weak neighbor of degree 3, and no E_2 -neighbor.*
 Then x has at most three other weak neighbors v_2, v_3 and v_4 of degree at most 5, which are by assumption of degree 5. Vertex x has no E_2 -neighbor, so they are E_3 or E_4 -neighbors of x . So x gives 1 to v_1 by R_3 , at most $\frac{1}{3}$ to v_2, v_3 and v_4 by R_7 or R_8 .
- (h) *Vertex x has no weak neighbor of degree 3.*
 Then x has at most four neighbors v_1, v_2, v_3 and v_4 of degree at most 5 that are either weak with degree at least 4 or semi-weak with degree 3. So x gives at most $\frac{1}{2}$ to each by R_4, R_5, R_6, R_7 or R_8 .

Consequently, after application of the discharging rules, every vertex and every face of G has a non-negative weight, $6|E| - 6|V| - 6|F| = (2|E| - 6|V|) + (4|E| - 6|F|) = \sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) \geq 0$, a contradiction to Euler's Formula. \square

6 Proof of Theorem 5

Let G be a minimal planar graph with $\Delta(G) \leq 8$ such that G is not 9-edge-choosable. By Lemma 4, graph G cannot contain (C_1) to (C_{11}) . Lemma 5 implies that G is a stable set, thus 9-edge-choosable,

a contradiction. □

7 Conclusion

The key idea in the proof lies in some recoloring arguments using directed graphs (see Claims 3, 4 and 6 for occurrences of it in the proof). It allowed us to deal with configurations that would not yield under usual techniques, and thus to improve Theorem 4. Though this simple argument does not seem to be enough to prove Conjecture 2 for $\Delta = 7$, it might be interesting to try to improve similarly Theorem 3.

Note that the proof could easily be adapted to prove that planar graphs with $\Delta \geq 8$ are $(\Delta + 1)$ -edge-choosable. This would however be of little interest considering the simple proof for $\Delta \geq 9$ presented in [6].

Conjecture 2 remains open for $\Delta = 5, 6$ and 7. It might be interesting to weaken the conjecture and ask whether all planar graphs are $(\Delta + 2)$ -edge-choosable. This is true for planar graphs with $\Delta \geq 7$ by Theorems 4 and 5. What about planar graphs with $\Delta = 6$?

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